MATH 4330/5330, Fourier Analysis
Section 5, The Dirichlet Kernel
Reformulating Fourier's Theorem

What does Fourier's Theorem really say? It says that, if f is a square-integrable function on $[0, 1)$, x is a point in $[0, 1)$, and for each integer n, c_n is defined by

$$c_n \equiv \widehat{f}(n) = \int_0^1 f(t)e^{-2\pi int}\,dt,$$

then

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx} \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{2\pi inx} \\ &= \lim_{N\to\infty} S_N(x), \end{aligned}$$

where S_N denotes the Nth partial sum of the Fourier series for f.

It is difficult to see how to prove this claim as it stands. The point of this section is to replace this limit of partial sums by a different kind of limit, one that hopefully is easier to attack. So, we want to express this claim of Fourier's in a different mathematical way. Namely,

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx} \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{2\pi inx} \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} \int_0^1 f(t)e^{-2\pi int}\,dt e^{2\pi inx} \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} \int_0^1 f(t)e^{2\pi in(x-t)}\,dt \\ &= \lim_{N\to\infty} \int_0^1 f(t) \sum_{n=-N}^{N} e^{2\pi in(x-t)}\,dt \\ &= \lim_{N\to\infty} \int_0^1 f(t)D_N(x-t)\,dt, \end{aligned}$$

where D_N is the function given by

$$D_N(t) = \sum_{n=-N}^{N} e^{2\pi int}.$$

DEFINITION. The sequence $\{D_N\}$ of functions defined above is called the *Dirichlet kernel.*

REMARK. Note that the calculation above relates the convergence of an infinite series to the limit of a certain sequence of integrals. This is important to mathematicians, because working with and estimating integrals is ordinarily much easier than the corresponding problems with infinite series.

It is also evident from the computation above that understanding this Dirichlet kernel is central to understanding Fourier's Theorem.

We begin with some initial observations about this kernel.

THEOREM 5.1. *The Dirichlet kernel satisfies the following:*

(1) $D_N(t) = \sum_{n=-N}^{N} e^{2\pi int}$.
(2) $D_N(t) = 1 + 2\sum_{k=1}^{N} \cos(2\pi kt)$.
(3) $D_N(0) = D_N(1) = 2N+1$, *and for* $0 < t < 1$,

$$D_N(t) = \frac{\sin(2\pi(N+\frac{1}{2})t)}{\sin(\pi t)}.$$

PROOF. The first formula is just the definition of D_N. The second follows directly from Euler's Theorem. (See the exercise below.)

Let us prove the third formula. We will use the formula for the sum of a geometric progression:

$$\sum_{n=0}^{r} z^n = \frac{1-z^{r+1}}{1-z}.$$

(See the exercise below.) Hence,

$$\begin{aligned}
D_N(t) &= \sum_{n=-N}^{N} e^{2\pi int} \\
&= e^{-2\pi iNt} \sum_{n=0}^{2N} e^{2\pi int} \\
&= e^{-2\pi iNt} \sum_{n=0}^{2N} (e^{2\pi it})^n \\
&= (e^{2\pi it})^{-N} \frac{1-(e^{2\pi it})^{2N+1}}{1-e^{2\pi it}} \\
&= \frac{(e^{2\pi it})^{-N} - (e^{2\pi it})^{N+1}}{1-e^{2\pi it}} \\
&= \frac{e^{\pi it}(e^{-2\pi iNt-\pi it} - e^{2\pi iNt+\pi it}}{e^{\pi it}(e^{-\pi it} - e^{\pi it}} \\
&= \frac{e^{-(N+\frac{1}{2})2\pi it} - e^{(N+\frac{1}{2})2\pi it}}{e^{-\pi it} - e^{\pi it}} \\
&= \frac{-2i\sin((N+\frac{1}{2})2\pi t)}{-2i\sin(\pi t)} \\
&= \frac{\sin(2\pi(N+\frac{1}{2})t)}{\sin(\pi t)}.
\end{aligned}$$

EXERCISE 5.1. (a) Prove part (2) of the preceding theorem.

(b) Derive the formula for the sum of a geometric series.

HINT: Set $S_r = \sum_{n=0}^{r} z^n$, and observe the two facts that $S_{r+1} = S_r + z^{r+1}$, and $S_{r+1} = 1 + zS_r$.

(c) Show that, if $|z| < 1$, then

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

EXERCISE 5.2. (a) Prove that $\int_0^1 D_N(t)\,dt = 1$ for all N.

(b) If $t \neq 0$, what can you say about the numbers $D_N(t)$? Are they converging to 0? Is D_N an example of a Dirac Δ function?

(c) Compute $\int_0^1 D_N^2(t)\,dt$.

HINT: Write

$$D_N^2 = (\sum_{n=-N}^{N} e^{2\pi i n t})^2 = \sum_{n=-N}^{N} e^{2\pi i n t} \times \sum_{m=-N}^{N} e^{2\pi i m t}.$$

Multiply it out, and then integrate. You should get $2N+1$ for your answer.

(d) Show that $D_N(t) = D_N(1-t) = D_N(-t)$; i.e., D_N is an even function and is also symmetric about the number 1/2. That is, $D_N(\frac{1}{2}+t) = D_N(\frac{1}{2}-t)$.

The Riemann-Lebesgue Lemma

Perhaps the most important computation in the beginning of Fourier analysis is the following:

THEOREM 5.2. (Riemann-Lebesgue Lemma) Let f be a continuous function on a closed interval $[a,b]$. Suppose that f is differentiable on the open interval (a,b) and that the derivative f' is bounded, i.e., $|f'(t)| \leq M$ for all $t \in (a,b)$. Then, for any sequence $\{k_n\}$ of numbers diverging to infinity, we have

$$\lim_{n\to\infty} \int_a^b f(t)\sin(k_n t)\,dt = 0.$$

REMARK. In fact, if f is any integrable function on $[a,b)$, then the same is true:

$$\lim_{n\to\infty} \int_a^b f(t)\sin(k_n t)\,dt = 0.$$

This stronger assertion of the Riemann-Lebesgue Lemma can be shown, though not easily, by proving that every integrable function is a kind of limit of differentiable functions, and then taking limits. We omit that argument here. Take a course in measure theory!

EXERCISE 5.3. (a) Prove the above theorem. Just integrate by parts, and use the fact that $|\int_a^b f'(t)\cos(k_n t)\,dt| \leq M(b-a)$.

(b) Let f be as in the Riemann-Lebesgue Lemma. As a special case of that lemma, note that

$$\lim_{N\to\infty} \int_a^b f(t)\sin(2\pi(N+\frac{1}{2})t)\,dt = 0.$$

Here is another, more subtle, property of the Dirichlet kernel.

THEOREM 5.3. *For any $0 < \delta < 1/2$, we have*

$$\lim_{N\to\infty} \int_0^{\delta} D_N(t)\,dt = \frac{1}{2}.$$

PROOF. Notice first that

$$\lim_{N\to\infty} \int_{\delta}^{1-\delta} D_N(t)\,dt = 0$$

by the Riemann-Lebesgue Lemma. Just take $[a,b]$ to be $[\delta, 1-\delta]$, and take $f(t) = 1/\sin(\pi t)$. Then apply part (b) of the preceding exercise.

Hence,

$$\lim_{N\to\infty} (\int_0^{\delta} D_N(t)\,dt + \int_{1-\delta}^{1} D_N(t)\,dt) = 1.$$

(Why?) Finally, since $D_N(t) = D_N(1-t)$, the result follows by replacing t by $1-t$ in the second integral.

EXERCISE 5.4. Let $0 < a < b < 1$, and let f be an integrable function on $[a,b]$. Show that

$$\lim_{N\to\infty} \int_a^b f(t) D_N(t)\,dt = 0.$$

Use the strong form of the Riemann-Lebesgue Lemma and the fact that $(1/\sin(\pi t))\times f(t)$ is integrable on $[a,b]$.

We finish this section with one final application of the Riemann-Lebesgue Lemma, which will then give us an important computation. The next exercise will be helpful here as well as later on.

EXERCISE 5.5. Let h be defined on the interval $(0, 1/2]$ by

$$h(t) = \frac{1}{\pi t} - \frac{1}{\sin(\pi t)}.$$

(a) Prove that $\lim_{t\to 0} h(t) = 0$, so that, defining $h(0)$ to be this limit, means that h is a continuous function on the closed interval $[0, 1/2]$.
HINT: Write

$$h(t) = \frac{1}{\pi t} - \frac{1}{\sin(\pi t)} = \frac{\sin(\pi t) - \pi t}{\pi t \sin(\pi t)},$$

and use L'Hopital's Rule two times.

(b) Prove that h is differentiable on $(0, 1/2)$ and that the derivative h' of h is bounded. (The only possible problem would be the limit of $h'(t)$ as t approaches 0. Do some algebra and use L'Hopital's Rule as in part (a).)

(c) Show that, for any $0 < \delta \leq 1/2$,

$$\lim_{B\to\infty} \int_0^{\delta} h(t) \sin(Bt)\,dt = 0.$$

THEOREM 5.4. *For any $0 < \delta \leq 1/2$, we have*

$$\lim_{N\to\infty} \int_0^\delta \frac{\sin(2\pi(N+\frac{1}{2})t)}{\pi t}\, dt = \frac{1}{2}.$$

REMARK. This is just like the preceding theorem, except we have replaced the $\sin(\pi t)$ in the denominator by the (simpler) term πt.

PROOF. Let h be the function of the preceding exercise. Note that h is continuous on the interval $[0, \delta]$, differentiable on the open interval $(0, \delta)$, and h' is bounded on $(0, \delta)$. We have

$$\begin{aligned}
\int_0^\delta \frac{\sin(2\pi(N+\frac{1}{2})t)}{\pi t}\, dt &= \int_0^\delta \sin(2\pi(N+\frac{1}{2})t)\frac{1}{\pi t}\, dt \\
&= \int_0^\delta \sin(2\pi(N+\frac{1}{2})t)(\frac{1}{\pi t} - \frac{1}{\sin(\pi t)} + \frac{1}{\sin(\pi t)})\, dt \\
&= \int_0^\delta \sin(2\pi(N+\frac{1}{2})t)(h(t) + \frac{1}{\sin(\pi t)})\, dt \\
&= \int_0^\delta h(t)\sin(2\pi(N+\frac{1}{2})t)\, dt + \int_0^\delta D_N(t)\, dt.
\end{aligned}$$

So, using part (c) of the preceding exercise, we obtain

$$\lim_{N\to\infty} \int_0^\delta \frac{\sin(2\pi(N+\frac{1}{2})t)}{\pi t} = 0 + \lim_{N\to\infty} \int_0^\delta D_N(t)\, dt = \frac{1}{2},$$

as desired.

Next, we present another important and famous computation.

THEOREM 5.5. *The improper Riemann integral*

$$\int_0^\infty \frac{\sin t}{t}\, dt = \lim_{B\to\infty} \int_0^B \frac{\sin t}{t}\, dt = \frac{\pi}{2}.$$

PROOF. Recall that we may evaluate the limit as B tends to ∞ by choosing any sequence $\{k_N\}$ that diverges to ∞ and then evaluating

$$\lim_{N\to\infty} \int_0^{k_N} \frac{\sin t}{t}\, dt.$$

We will use the sequence $k_N = \pi(N + \frac{1}{2})$.

From the preceding theorem, we know that

$$\frac{\pi}{2} = \lim_{N\to\infty} \int_0^{1/2} \frac{\sin(2\pi(N+\frac{1}{2})t)}{t}\, dt.$$

So, substituting $t = s/(2\pi(N + \frac{1}{2})$, we get

$$\frac{\pi}{2} = \lim_{N\to\infty} \int_0^{\pi(N+\frac{1}{2})} \frac{\sin s}{s}\, ds,$$

which proves the claim.