MATH 4330/5330, Fourier Analysis
Section 7, L^2 convergence of Fourier Series

In the last section we investigated the convergence of the Fourier series for a function f at a single point x, so-called pointwise convergence. We next study the convergence of Fourier series relative to a kind of average behavior. This kind of convergence is called L^2 convergence or *convergence in mean*.

DEFINITION. A sequence $\{f_n\}$ of periodic, square-integrable functions is said to *converge in* L^2 to a function f if the sequence of numbers $\{\int_0^1 |f_n(x) - f(x)|^2\, dx\}$ converges to 0.

EXERCISE 7.1. For each $n \geq 1$ define a function f_n on $[0,1)$ as follows: $f_n(x) = \sqrt{n}x^n$.

(a) Show that the sequence $\{f_n(x)\}$ converges to 0 for every $0 \leq x < 1$. That is, show that

$$\lim_{n\to\infty} \sqrt{n}x^n = 0.$$

(b) Compute

$$\int_0^1 |f_n(x) - 0|^2\, dx = \int_0^1 nx^{2n}\, dx,$$

and verify that this sequence does not converge to 0.

(c) Conclude that, just because a sequence $\{f_n\}$ converges pointwise, it does not mean that it must converge in L^2.

EXERCISE 7.2. Now let f_n be defined by $f_n(x) = n^{1/3}$ if $0 \leq x \leq 1/n$, and $f_n(x) = 0$ otherwise.

(a) Sketch the graph of f_n.

(b) Show that $f_n(0) = n^{1/3}$, which does not converge to anything.

(c) Compute

$$\int_0^1 |f_n(x) - 0|^2\, dx = \int_0^{\frac{1}{n}} n^{2/3} = n^{-\frac{1}{3}}.$$

Conclude that the sequence $\{f_n\}$ converges to the 0 function in L^2. Conclude then that, just because a sequence $\{f_n\}$ converges in L^2, it need not converge pointwise.

Here is Fourier's Theorem in this L^2 convergence context. It is **perfect.**

THEOREM 7.1. *Let f be a periodic, square-integrable function. Then the Fourier series for f converges in L^2 to f; i.e.,*

$$\lim_{N\to\infty} \int_0^1 |S_N(x) - f(x)|^2\, dx = 0.$$

The proof of this theorem will have to wait until we have developed some more techniques.

For simplicity of notation, we will write ϕ_n for the exponential function $\phi_n(x) = e^{2\pi inx}$.

EXERCISE 7.3. Verify that, in the ϕ_n notation, we have the following expressions for Fourier coefficients and Fourier series.

$$\widehat{f}(n) = \int_0^1 f(x)\overline{\phi_n(x)}\, dx = \int_0^1 f(x)\phi_{-n}(x)\, dx$$

and

$$S_N(x) = \sum_{n=-N}^{N} \widehat{f}(n)\phi_n(x).$$

PROPOSITION 7.2. *The collection of functions $\{\phi_n\}$ satisfies the following properties:*

(1) *For every integer n, $\int_0^1 |\phi_n(x)|^2\,dx = 1$.*
(2) *If $n \neq k$, then $\int_0^1 \phi_n(x)\overline{\phi_k(x)}\,dx = 0$.*
(3) *For any integers n and k, $\int_0^1 \phi_n(x)\overline{\phi_k(x)}\,dx = \delta_{n,k}$, where $\delta_{n,k}$ is the Kronecker δ function defined by $\delta_{n,k} = 0$ if $n \neq k$ and $= 1$ if $n = k$.*
(4) *If $f = \sum_{n=-N}^{N} c_n\phi_n$ is a finite linear combination of the ϕ_n's, then*

$$\int_0^1 |f(x)|^2\,dx = \sum_{n=-N}^{N} |c_n|^2.$$

PROOF. We leave the proof of parts (1), (2), and (3) to the next exercise. Rather, let us use part (3) to prove part (4). Hence, suppose $f = \sum_{n=-N}^{N} c_n\phi_n$. Then

$$\begin{aligned}
\int_0^1 |f(x)|^2\,dx &= \int_0^1 f(x)\overline{f(x)}\,dx \\
&= \int_0^1 \sum_{n=-N}^{N} c_n\phi_n(x) \times \overline{\sum_{k=-N}^{N} c_k\phi_k(x)}\,dx \\
&= \sum_{n=-N}^{N}\sum_{k=-N}^{N} \int_0^1 c_n\overline{c_k}\phi_n(x)\overline{\phi_k(x)}\,dx \\
&= \sum_{n=-N}^{N}\sum_{k=-N}^{N} c_n\overline{c_k} \int_0^1 \phi_n(x)\overline{\phi_k(x)}\,dx \\
&= \sum_{n=-N}^{N}\sum_{k=-N}^{N} c_n\overline{c_k}\delta_{n,k} \\
&= \sum_{n=-N}^{N} c_n\overline{c_n} \\
&= \sum_{n=-N}^{N} |c_n|^2,
\end{aligned}$$

as desired.

EXERCISE 7.4. (a) Prove parts (1), (2), and (3) of the preceding proposition.

(b) Let f be a square-integrable function on $[0, 1)$, and write S_N for the Nth partial sum of its Fourier series. Use part (4) of the preceding proposition to show that

$$\int_0^1 |S_N(x)|^2\,dx = \sum_{n=-N}^{N} |\widehat{f}(n)|^2.$$

The next result is famous, and its proof is tricky.

THEOREM 7.3. (Bessel's Inequality) Let f be a periodic, square-integrable function, and write $\widehat{f}$ for its Fourier transform. Then, for every N, we have

$$\sum_{n=-N}^{N} |\widehat{f}(n)|^2 \leq \int_0^1 |f(x)|^2 \, dx.$$

Consequently,

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \leq \int_0^1 |f(x)|^2 \, dx.$$

PROOF. Fix N, and as usual write $S_N(x) = \sum_{n=-N}^{N} \widehat{f}(n)\phi_n(x)$ for the partial sums of the Fourier series for f. Then consider the following calculation:

$$
\begin{aligned}
0 &\leq \int_0^1 |f(x) - S_N(x)|^2 \, dx \\
&= \int_0^1 (f(x) - S_N(x))\overline{(f(x) - S_N(x))} \, dx \\
&= \int_0^1 (f(x) - S_N(x))(\overline{f(x)} - \overline{S_N(x)}) \, dx \\
&= \int_0^1 f(x)\overline{f(x)} - \int_0^1 f(x)\overline{S_N(x)} \, dx \\
&\qquad - \int_0^1 S_N(x)\overline{f(x)} \, dx + \int_0^1 S_N(x)\overline{S_N(x)} \, dx \\
&= \int_0^1 |f(x)|^2 \, dx - \int_0^1 f(x) \sum_{n=-N}^{N} \overline{\widehat{f}(n)}\overline{\phi_n(x)} \, dx \\
&\qquad - \int_0^1 \sum_{n=-N}^{N} \widehat{f}(n)\phi_n(x)\overline{f(x)} \, dx + \int_0^1 |S_N(x)|^2 \, dx \\
&= \int_0^1 |f(x)|^2 \, dx - \sum_{n=-N}^{N} \overline{\widehat{f}(n)} \int_0^1 f(x)\overline{\phi_n(x)} \, dx \\
&\qquad - \sum_{n=-N}^{N} \widehat{f}(n) \int_0^1 \phi_n(x)\overline{f(x)} \, dx + \sum_{n=-N}^{N} |\widehat{f}(n)|^2 \\
&= \int_0^1 |f(x)|^2 \, dx - \sum_{n=-N}^{N} \overline{\widehat{f}(n)}\widehat{f}(n) \\
&\qquad - \sum_{n=-N}^{N} \widehat{f}(n)\overline{\int_0^1 f(x)\overline{\phi_n(x)} \, dx} + \sum_{n=-N}^{N} |\widehat{f}(n)|^2 \\
&= \int_0^1 |f(x)|^2 \, dx - \sum_{n=-N}^{N} |\widehat{f}(n)|^2 \\
&\qquad - \sum_{n=-N}^{N} |\widehat{f}(n)|^2 + \sum_{n=-N}^{N} |\widehat{f}(n)|^2 \\
&= \int_0^1 |f(x)|^2 \, dx - \sum_{n=-N}^{N} |\widehat{f}(n)|^2,
\end{aligned}
$$

showing that

$$
0 \leq \int_0^1 |f(x)|^2 - \sum_{n=-N}^{N} |\widehat{f}(n)|^2,
$$

from which the first assertion of the theorem follows.

The second assertion follows immediately from the fact that the infinite sum is the limit of the partial sums.

DEFINITION. By $L^2(\mathbb{Z})$, or simply l^2, we mean the set of all sequences $\{c_n\}_{-\infty}^{\infty}$ for which $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$. Such sequences are called *square-summable.*

EXERCISE 7.5. Show that if $f \in L^2([0,1))$, then $\widehat{f} \in l^2$.

The preceding exercise tells us something about the range of the Fourier transform. It says that if f is square-integrable, then $\widehat{f}$ is square-summable. The next theorem is an even more clean result about the range of the transform.

THEOREM 7.4. *The range of the Fourier transform on L^2 is all of l^2. That is, if $\{c_n\}$ is a square-summable sequence in l^2, then there exists a periodic, square-integrable function g such that $c_n = \widehat{g}(n)$ for all n. Moreover,*

$$g(x) = \sum_{n=-\infty}^{\infty} c_n \phi_n(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n \phi_n m(x),$$

where the limit is taken in the L^2 sense.

REMARK. The proof of this theorem is a bit too advanced for this course. The argument goes like this: First show that if $T_N(x)$ is defined to be $\sum_{n=-N}^{N} c_n\phi_n(x)$, then the sequence $\{T_N\}$ converges in L^2 to some function g. (The fact that the sequence $\{c_n\}$ is square-summable is what makes this so.) Then we show that $\widehat{g}(n)$ must equal c_n for all n. This part depends on some advanced continuity notions with respect to L^2 convergence. We will just have to accept this theorem for the moment.

We can now, with the help of the preceding result, prove Fourier's Theorem for L^2 convergence.

PROOF OF THEOREM 7.1. Let f be a periodic, square-integrable function. Then, by Bessel's Inequality, or more specifically Exercise 7.5, we see that the sequence $\{c_n\} \equiv \{\widehat{f}(n)\}$ is square-summable. So, by Theorem 7.4, there exists a periodic, square-integrable function g such that $g = \sum_{n=-\infty}^{\infty} c_n\phi_n$. Moreover, $\widehat{g}(n) = c_n$ for all n.

But this means that $\widehat{f}(n) = c_n = \widehat{g}(n)$ for all n. That is, $\widehat{f} = \widehat{g}$. Since we have seen that the Fourier transform is 1-1, this implies that $f = g$, or that

$$f = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\phi_n$$

in L^2. This is exactly the claim in Theorem 7.1.

Bessel's Inequality gives an inequality between $\int |f(x)|^2\,dx$ and the infinite series $\sum |\widehat{f}(n)|^2$. Actually, this inequality turns out to be a precise equality.

THEOREM 7.5. (Parseval's Equality) Let f be a periodic, square-integrable function. Then

$$\int_0^1 |f(x)|^2\,dx = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2.$$

PROOF. There is one serious mathematical point in this argument. See if you can spot it!

$$\begin{aligned}
\int_0^1 |f(x)|^2\,dx &= \int_0^1 f(x)\overline{f(x)}\,dx \\
&= \int_0^1 f(x)\overline{\lim_{N\to\infty} S_N(x)}\,dx \\
&= \int_0^1 f(x)\lim_{N\to\infty} \overline{S_N(x)}\,dx \\
&= \lim_{N\to\infty}\int_0^1 f(x)\overline{\sum_{n=-N}^{N} \widehat{f}(n)\phi_n(x)}\,dx \\
&= \lim_{N\to\infty}\int_0^1 f(x)\sum_{n=-N}^{N} \overline{\widehat{f}(n)}\,\overline{\phi_n(x)}\,dx \\
&= \lim_{N\to\infty}\sum_{n=-N}^{N} \overline{\widehat{f}(n)}\int_0^1 f(x)\overline{\phi_n(x)}\,dx \\
&= \lim_{N\to\infty}\sum_{n=-N}^{N} \overline{\widehat{f}(n)}\widehat{f}(n) \\
&= \lim_{N\to\infty}\sum_{n=-N}^{N} |\widehat{f}(n)|^2 \\
&= \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2.
\end{aligned}$$

EXERCISE 7.6. (a) Use the function $f(x) = 1/2 - x$ and Parseval's Equality to derive the formula

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(b) Can you recall or figure out a function f whose Fourier coefficients satisfy something like $\widehat{f}(n) = c/n^2$. Use this function and Parseval's Equality to compute $\sum_{n=1}^{\infty} 1/n^4$. (You should get $\pi^4/90$.