MATH 4330/5330, Fourier Analysis
Section 9
Properties of the Fourier Transform

The reason that the Fourier transform is so useful is that it converts operations on functions f into different operations on the functions $\widehat{f}$. We must try to catalog how these conversions work. The next two results generalize Theorem 8.6.

PROPOSITION 9.1. (Translation)

(1) If f is a periodic, square-integrable function, and if a is any real number, define f_a to be the function given by $f_a(x) = f(x+a)$. Then

$$\widehat{f_a}(n) = e^{2\pi i n a}\widehat{f}(n).$$

(2) If f is an element of $L^1(\mathbb{R}$, and a is any real number, define f_a to be the function given by $f_a(x) = f(x+a)$. Then

$$\widehat{f_a}(\omega) = e^{2\pi i \omega a}\widehat{f}(\omega).$$

In both cases, the translation operator is converted into a multiplication operator.

EXERCISE 9.1. Prove Proposition 9.1. It's just a matter of changing variables in the integral.

PROPOSITION 9.2. (Differentiation)

(1) Suppose f is a differentiable function, and assume that both f and f' are periodic and square-integrable. Then

$$\widehat{f'}(n) = 2\pi i n\widehat{f}(n).$$

(2) Suppose f is a differentiable function on $\mathbb{R}$, and assume that both f and f' are elements of $L^1(\mathbb{R})$. Then

$$\widehat{f'}(\omega) = 2\pi i \omega\widehat{f}(\omega).$$

In both cases, the differentiation operator is converted into a multiplication operator.

EXERCISE 9.2. (a) Prove Proposition 9.2. (Just integrate by parts.)

(b) Generalize Proposition 9.2 to compute the Fourier transform of the second derivative of f. What about higher derivatives?

Convolution of Functions

DEFINITION. Let f and g be two periodic, square-integrable functions, and define another function, denoted $f * g$, by

$$f * g(x) = \int_0^1 f(x-t)g(t)\,dt.$$

We call the function $f * g$ the *convolution* of f and$g$.

EXERCISE 9.3. (a) Can you explain the minus sign?

(b) If f and g are periodic, square-integrable functions, show that $f * g$ is also periodic.

(c) Let f be a periodic, square-integrable function, and write $\{S_N\}$ for the sequence of partial sums of its Fourier series. Prove that $S_N = f * D_N$. That is, the partial sum S_N is the convolution of the function f with the Dirichlet kernel D_N.

(d) As usual, let $\phi_n(x) = e^{2\pi inx}$. Compute the convolution $\phi_n * \phi_k$.

(e) Let f be the periodic function defined on $[0,1)$ by $f(x) = x$. Compute the convolution $f * f$.

THEOREM 9.3. *The convolution of square-integrable, periodic functions has the following properties:*

1. *Convolution is a binary operation on the set of periodic, square-integrable functions.*
2. *The convolution of f and g is always a bounded function.*
3. $f * g = g * f$, *i.e., the convolution is commutative.*
4. $f * (g * h) = (f * g) * h$, *i.e., the convolution is associative.*
5. $f * (g + h) = f * g + f * h$, *i.e., convolution is distributive over addition.*
6. $\|f * g\| \leq \|f\| \|g\|$, *where the norm of a function f is given by*

$$\|f\| = \sqrt{\int_0^1 |f(x)|^2 \, dx}.$$

PROOF. We prove parts (4) and (6), leaving the rest to the exercise that follows.

Let $f, g,$ and h be periodic and square-integrable. Find the subtle mathematical step in the following.

$$\begin{aligned}
f * (g * h)(x) &= \int_0^1 f(x-t) g * h(t) \, dt \\
&= \int_0^1 f(x-t) \int_0^1 g(t-s) h(s) \, ds dt \\
&= \int_0^1 \int_0^1 f(x-t) g(t-s) h(s) \, ds dt \\
&= \int_0^1 \int_0^1 f(x-t) g(t-s) h(s) \, dt ds \\
&= \int_0^1 \int_0^1 f(x-t-s) g(t) h(s) \, dt ds \\
&= \int_0^1 \int_0^1 f(x-s-t) g(t) \, dt h(s) \, ds \\
&= \int_0^1 f * g(x-s) h(s) \, ds \\
&= (f * g) * h(x),
\end{aligned}$$

and this proves (4).

Let us verify part (6). Thus let f and g be periodic, square-integrable functions, and for each x write h_x for the function defined by $h_x(t) = \overline{f(x-t)}$. We will need this notation in the calculation below, and also, at some point we will use the Cauchy-Schwarz Inequality. See if you can spot it.

$$
\begin{aligned}
\|f * g\|^2 &= \int_0^1 |f * g(x)|^2 \, dx \\
&= \int_0^1 | \int_0^1 f(x-t)g(t) \, dt|^2 \, dx \\
&= \int_0^1 | \int_0^1 g(t)\overline{h_x}(t) \, dt|^2 \, dx \\
&= \int_0^1 |\langle g \mid h_x \rangle|^2 \, dx \\
&\leq \int_0^1 \|g\|^2 \|h_x\|^2 \, dx \\
&= \|g\|^2 \int_0^1 \|h_x\|^2 \, dx \\
&= \|g\|^2 \int_0^1 \int_0^1 |f(x-t)|^2 \, dt dx \\
&= \|g\|^2 \int_0^1 \int_0^1 |f(t)|^2 \, dt dx \\
&= \|g\|^2 \int_0^1 \|f\|^2 \, dx \\
&= \|g\|^2 \|f\|^2,
\end{aligned}
$$

which proves the desired inequality by taking square roots.

EXERCISE 9.4. Prove parts (1), (2), (3), and (5) of the preceding theorem. Can you explain the minus sign now?

REMARK. The set $L^2([0,1))$ is now an example of a *normed algebra.* The properties in the preceding theorem are just what's required to be a normed algebra. It is a vector space, and in addition it has a notion of multiplication. Finally, both addition and multiplication are related to the norm by inequalities. Many questions come up now about this convolution operation. Is there an identity? That is, is there a function e such that $f = f * e$ for every f. Are there lots of functions that satisfy $f * f = f$? Can we solve polynomial equations in L^2? That is, if $p(x) = \sum_{j=0}^n c_j x^j$ is a polynomial, is there a function f in L^2 such that

$$
p(f) = \sum_{j=0}^n c_j f^j = 0?
$$

Of course, here we mean by f^j the convolution product of f with itself j times.

THEOREM 9.4. (Convolution Theorem) Let f and g be periodic, square-integrable functions. Then

$$
\widehat{f * g} = \widehat{f}\widehat{g},
$$

or more explicitly,

$$
\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)
$$

for all n. Under the Fourier transform, the binary operation of convolution is converted into the ordinary product of functions.

PROOF. We just compute. Watch for the mathematical subtleties.

$$\begin{aligned}
\widehat{f*g}(n) &= \int_0^1 f*g(x)e^{-2\pi inx}\,dx \\
&= \int_0^1 \int_0^1 f(x-t)g(t)\,dt e^{-2\pi inx}\,dx \\
&= \int_0^1 \int_0^1 f(x-t)g(t)e^{-2\pi inx}\,dtdx \\
&= \int_0^1 \int_0^1 f(x-t)g(t)e^{-2\pi inx}\,dxdt \\
&= \int_0^1 \int_0^1 f(x)g(t)e^{-2\pi in(x+t)}\,dxdt \\
&= \int_0^1 \int_0^1 f(x)e^{-2\pi inx}\,dx g(t)e^{-2\pi int}\,dt \\
&= \int_0^1 \widehat{f}(n)g(t)e^{-2\pi int}\,dt \\
&= \widehat{f}(n)\widehat{g}(n),
\end{aligned}$$

as desired.

Convolution on the real line

DEFINITION. Let f and g be two elements of $L^1(\mathbb{R})$, and define a third function denoted $f*g$, by

$$f*g(x) = \int_{\mathbb{R}} f(x-t)g(t)\,dt.$$

We call the function $f*g$ the *convolution* of f and$g$.

EXERCISE 9.5. (a) Let f be the function defined on the real line by

$$f(x) = \begin{cases} 1 & -\frac{1}{2} \le x < \frac{1}{2} \\ 0 & otherwise \end{cases}.$$

Compute the convolution $f*f$.

(b) Let f be defined by $f(x) = e^{-\pi x^2}$. Compute $f*f$. You may want to consult part (d) of Exercise 1.4.

(c) For each positive number t, let k_t be the function given by

$$k_t(x) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}.$$

Compute $k_t * k_s$. YOu should get k_{t+s}. (We have called the functions $\{k_t\}$ the heat kernel. Because of the calculation above, it is often called the *heat semigroup*.)

THEOREM 9.5. *The convolution of L^1 functions has the following properties:*

(1) *Convolution is a binary operation on $L^1(\mathbb{R})$.*
(2) $f * g = g * f$, *i.e., the convolution is commutative.*
(3) $f * (g * h) = (f * g) * h$, *i.e., the convolution is associative.*
(4) $f * (g + h) = f * g + f * h$, *i.e., convolution is distributive over addition.*
(5) $\|f * g\| \leq \|f\|\|g\|$, *where the norm of a function f is given this time as the L^1 norm:*

$$\|f\| = \int_{\mathbb{R}} |f(t)|\, dt.$$

PROOF. The proofs are analogous to the corresponding arguments given in the proof of Theorem 9.3.

REMARK. The set $L^1(\mathbb{R})$ is another example of a *normed algebra*. It is a vector space, and in addition it has a notion of multiplication. Finally, both addition and multiplication are related to the norm by inequalities. As before, many questions come up now about this convolution operation. Is there an identity? That is, is there a function e such that $f = f * e$ for every f. Are there lots of functions that satisfy $f * f = f$? Can we solve polynomial equations in L^2? That is, if $p(x) = \sum_{j=0}^{n} c_j x^j$ is a polynomial, is there a function f in $L^1(\mathbb{R})$ such that

$$p(f) = \sum_{j=0}^{n} c_j f^j = 0?$$

Of course, here we mean by f^j the convolution product of f with itself j times.

THEOREM 9.6. (Convolution Theorem) Let f and g be L^1 functions. Then

$$\widehat{f * g} = \widehat{f}\widehat{g},$$

or more explicitly,

$$\widehat{f * g}(\omega) = \widehat{f}(\omega)\widehat{g}(\omega)$$

for all ω. Under the Fourier transform, the binary operation of convolution is converted into the ordinary product of functions.

PROOF. We just compute. Watch for the mathematical subtleties.

$$\begin{aligned}
\widehat{f * g}(\omega) &= \int_{\mathbb{R}} f * g(x) e^{-2\pi i \omega x}\, dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t) g(t)\, dt e^{-2\pi i \omega x}\, dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t) g(t) e^{-2\pi i \omega x}\, dt dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t) g(t) e^{-2\pi i \omega x}\, dx dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) g(t) e^{-2\pi i \omega (x+t)}\, dx dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) e^{-2\pi i x \omega}\, dx g(t) e^{-2\pi i t \omega}\, dt \\
&= \int_{\mathbb{R}} \widehat{f}(\omega) g(t) e^{-2\pi i t \omega}\, dt \\
&= \widehat{f}(\omega)\widehat{g}(\omega),
\end{aligned}$$

as desired.

Let's Solve Some Problems!

1. The heat equation on the circle.

Let f be a periodic, square-integrable function, and suppose that $u(t,x)$ is a solution to the heat equation on the circle having initial value $f(x)$. Can we prove that such a u exists? Can we prove it is unique? Can we find out exactly what u is?

For each $t \geq 0$, write f_t for the function of x given by $f_t(x) = u(t,x)$. Note that $f_0(x) = u(0,x) = f(x)$. Then for every $t \geq 0$ f_t is periodic and square-integrable. So, we may write

$$\begin{aligned} u(t,x) &= f_t(x) \\ &= \sum_{n=-\infty}^{\infty} \widehat{f_t}(n) e^{2\pi i n x} \\ &= \sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi i n x}, \end{aligned}$$

where $c_n(t)$ is just another name for $\widehat{f_t}(n)$. (How do we know these formulas are justified?) In particular, $c_n(0) = \widehat{f_0}(n) = \widehat{f}(n)$.

So, assuming that a solution u exists, and writing $u(t,x)$ as $\sum c_n(t)e^{2\pi i n x}$, what is the derivative of u with respect to t? Be alert for the subtle mathematical steps here.

$$\begin{aligned} \frac{\partial u}{\partial t} u(t,x) &= \frac{d}{dt}\Big(\sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi i n x}\Big) \\ &= \sum_{n=-\infty}^{\infty} c_n'(t) e^{2\pi i n x}. \end{aligned}$$

And, what is the second derivative of u with respect to x?

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} u(t,x) &= \Big(\frac{d}{dx}\Big)^2\Big(\sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi i n x}\Big) \\ &= \sum_{n=-\infty}^{\infty} c_n(t)(-4\pi^2 n^2) e^{2\pi i n x}. \end{aligned}$$

Hence, for each $t > 0$, we an an equality between two Fourier series:

$$\sum_{n=-\infty}^{\infty} c_n'(t) e^{2\pi i n x} = \sum_{n=-\infty}^{\infty} c_n(t)(-4\pi^2 n^2) e^{2\pi i n x},$$

from which it follows (why?) that for each integer n, we have

$$c_n'(t) = c_n(t)(-4\pi^2 n^2).$$

Now we can solve this simple differential equation for the functions c_n, using Theorem 2.1 for instance. Indeed, we get that

$$c_n(t) = c_n(0) e^{-4\pi^2 n^2 t} = \widehat{f}(n) e^{-4\pi^2 n^2 t}.$$

Now, for every $t > 0$, the sequence $\{e^{-4\pi^2 n^2 t}\}$ is square-summable, and by Theorem 7.4 we may define a function k_t by $k_t(x) = \sum_n e^{-4\pi^2 n^2 t} e^{2\pi inx}$. Recall also that $e^{-4\pi^2 n^2 t} = \widehat{k_t}(n)$.

Finally, a formula for the solution to the heat equation is given by

$$\begin{aligned}
u(t,x) &= \sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi inx} \\
&= \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{-4\pi^2 n^2 t} e^{2\pi inx} \\
&= \sum_{n=-\infty}^{\infty} \widehat{f}(n) \widehat{k_t}(n) e^{2\pi inx} \\
&= \sum_{n=-\infty}^{\infty} \int_0^1 f(s) e^{-2\pi ins}\, ds \widehat{k_t}(n) e^{2\pi inx} \\
&= \sum_{n=-\infty}^{\infty} \int_0^1 f(s) e^{-2\pi ins} \widehat{k_t}(n) e^{2\pi inx}\, ds \\
&= \sum_{n=-\infty}^{\infty} \int_0^1 f(s) \widehat{k_t}(n) e^{2\pi in(x-s)}\, ds \\
&= \int_0^1 \sum_{n=-\infty}^{\infty} f(s) \widehat{k_t}(n) e^{2\pi in(x-s)}\, ds \\
&= \int_0^1 f(s) \sum_{n=-\infty}^{\infty} \widehat{k_t}(n) e^{2\pi in(x-s)}\, ds \\
&= \int_0^1 f(s) k_t(x-s)\, ds \\
&= f * k_t(x).
\end{aligned}$$

In this case, the parameterized family k_t of functions is called the *heat kernel* on the circle. Unfortunately, we do not have an explicit closed form expression for the heat kernel k_t in this case. Only the Fourier series for it is known.

EXERCISE 9.6. Verify that the argument above shows that, for any periodic, square-integrable, initial value function f, there exists a unique solution to the initial value problem for the heat equation on the circle. That is, show that the formula above for $u(t,x)$ really is a solution to the heat equation, and use the arguments above to show that any solution of the heat equation on the circle has to coincide with this one.

2. A Boundary Value Problem.

What conditions must real numbers a, b, and c satisfy in order that there be a nontrivial, periodic solution to the differential equation

$$af'' + bf' + cf = 0?$$

(Let's assume that $a \neq 0$ to avoid trivial cases.) That is, we want a solution f to this differential equation, and we want f to satisfy the *boundary value condition* $f(0) = f(1)$.

What is the transformed version of this differential equation? It is

$$-4\pi^2 n^2 a\widehat{f}(n) + 2\pi i n b\widehat{f}(n) + c\widehat{f}(n) = 0$$

for all n. Therefore, for every value of n for which $\widehat{f}(n) \neq 0$, we must have that the integer n is a root of the quadratic equation

$$-4\pi^2 a x^2 + 2\pi i b x + c = 0.$$

There can be at most two roots of this equation, So $\widehat{f}(n)$ can be nonzero for at most two integers n. Perhaps more interesting is what conditions on $a, b,$ and c will suffice to make a root of this quadratic equation be an integer. Using the quadratic formula, we have that the roots are

$$x = \frac{-2\pi i b \pm \sqrt{-4\pi^2 b^2 + 16\pi^2 ac}}{-8\pi^2 a}.$$

If a root of this quadratic equation is an integer, hence a real number, then, to be sure that there is no imaginary part of the root, either the coefficient b must be 0 or the coefficient c must be 0. In the latter case, i.e., when $c = 0$, the only real root will be 0. Hence, in that case, $\widehat{f}(n) = 0$ except when $n = 0$, and this means that f is a constant function.

Next, assuming that $b = 0$, we see that the solutions are

$$x = \pm\frac{4\pi\sqrt{ac}}{8\pi^2 a} = \pm\frac{1}{2\pi}\sqrt{\frac{c}{a}}.$$

So, the only time this solution would be an integer n is if $c/a = 4\pi^2 n^2$ for some n. Hence, the only time the differential equation $af'' + bf' + cf = 0$ has a periodic solution is when f is a constant function, or $b = 0$ and $c/a = 4\pi^2 n^2$, for some n. In this case, the differential equation becomes

$$f'' + 4\pi^2 n^2 f = 0,$$

and the solution is necessarily of the form

$$f(x) = \alpha e^{2\pi i n x} + \beta e^{-2\pi i n x}.$$

EXERCISE 9.7. (a) Use Fourier analysis to solve the inhomogeneous differential equation

$$f'' + f' + f = \sin(6\pi x).$$

Write down the transformed version of this differential equation, solve that equation for $\widehat{f}$, and then write down a formula for f.

(b) Use Fourier analysis to show that there exists a periodic solution to the inhomogeneous differential equation

$$5f''(x) + 4f'(x) + 3f(x) = frac12 - x.$$

3. Questions about Convolution.

Is there an identity for convolution. That is, is there a function $e \in L^2([0,1))$ for which $e * f = f$ for all f. Suppose there is. Then, on the transform side, we would have

$$\widehat{e}(n)\widehat{f}(n) = \widehat{f}(n)$$

for all f and all n. This would imply that $\widehat{e}(n) = 1$ for all n. (Why?) But this is impossible because of the Riemann-Lebesgue Lemma.

EXERCISE 9.8. Is there an identity for convolution of L^1 functions on the real line? That is, is thee an L^1 function e on $\mathbb{R}$ for which $f = e * f$ for all L^1 functions f?

Are there any square-integrable periodic functions f for which $f = f * f$? Transforming, we would ask are there any functions $\widehat{f}$ for which $\widehat{f}(n) = \widehat{f}(n) \times \widehat{f}(n) = \widehat{f}^2(n)$? This happens if and only if $\widehat{f}(n)$ is 0 or 1 for every n. There are lots of such functions; any function whose Fourier coefficients are 0 or 1. What are some examples? Every ϕ_n works. The Dirichlet kernel itself works. $2\cos(2\pi kx)$ for any nonzero k works.

EXERCISE 9.9. Are there any L^1 functions f on the real line such that $f = f * f$?

Do any square-integrable periodic functions f have square roots with respect to convolution? That is, are there any functions f for which $f = g * g$ for some g?

On the transform side, we would be asking if there are any functions $\widehat{f}$ such that $\widehat{f} = \widehat{g}^2$; i.e., $\widehat{f}(n) = \widehat{g}^2(n)$ for all n. Well, these are just complex numbers, each of which has a square root. So, given a periodic, square-integrable function f, Why not define

$$g(x) = \sum_{n=-\infty}^{\infty} \sqrt{\widehat{f}(n)} e^{2\pi i n x}?$$

Is this g a periodic, square-integrable function? When can we define a function by means of a sequence $\{c_n\}$? Again, think about Section 7, particularly Theorem 7.4.

EXERCISE 9.10. If $p(x) = \sum_{j=0}^{n} c_j x^j$ is a polynomial, when is there a function f in L^2 such that

$$p(f) = \sum_{j=0}^{n} c_j f^j = 0?$$

Of course, here we mean by f^j the convolution product of f with itself j times.