CHAPTER III

TOPOLOGICAL VECTOR SPACES AND CONTINUOUS LINEAR FUNCTIONALS

The marvelous interaction between linearity and topology is introduced in this chapter. Although the most familiar examples of this interaction may be normed linear spaces, we have in mind here the more subtle, and perhaps more important, topological vector spaces whose topologies are defined as the weakest topologies making certain collections of functions continuous. See the examples in Exercises 3.8 and 3.9, and particularly the Schwartz space $\mathcal{S}$ discussed in Exercise 3.10.

DEFINITION. A *topological vector space* is a real (or complex) vector space X on which there is a Hausdorff topology such that:

(1) The map $(x, y) \to x+y$ is continuous from $X \times X$ into X. (Addition is continuous.) and

(2) The map $(t, x) \to tx$ is continuous from $\mathbb{R} \times X$ into X (or $\mathbb{C} \times X$ into X). (Scalar multiplication is continuous.)

We say that a topological vector space X is a *real* or *complex* topological vector space according to which field of scalars we are considering. A complex topological vector space is obviously also a real topological vector space.

A metric d on a vector space X is called *translation-invariant* if $d(x+z, y+z) = d(x, y)$ for all $x, y, z \in X$. If the topology on a topological vector space X is determined by a translation-invariant metric d, we call X (or (X, d)) a *metrizable* vector space. If x is an element of a metrizable vector space (X, d), we denote by $B_\epsilon(x)$ the ball of radius

ϵ around x; i.e., $B_\epsilon(x) = \{y : d(x,y) < \epsilon\}$. If the topology on a vector space X is determined by the translation-invariant metric d defined by a norm on X, i.e., $d(x,y) = \|x-y\|$, we call X a *normable* vector space. If the topology on X is determined by some complete translation-invariant metric, we call X a *Fréchet* space.

The topological vector space X is called *separable* if it contains a countable dense subset.

Two topological vector spaces X_1 and X_2 are *topologically isomorphic* if there exists a linear isomorphism T from X_1 onto X_2 that is also a homeomorphism. In this case, T is called a *topological isomorphism.*

EXERCISE 3.1. (a) Let X be a topological vector space, and let x be a nonzero element of X. Show that the map $y \to x+y$ is a (nonlinear) homeomorphism of X onto itself. Hence, U is a neighborhood of 0 if and only if $x+U$ is a neighborhood of x. Show further that if U is an open subset of X and S is any subset of X, then $S+U$ is an open subset of X.

(b) Show that $x \to -x$ is a topological isomorphism of X onto itself. Hence, if U is a neighborhood of 0, then $-U$ also is a neighborhood of 0, and hence $V = U \cap (-U)$ is a *symmetric* neighborhood of 0; i.e., $x \in V$ if and only if $-x \in V$.

(c) If U is a neighborhood of 0 in a topological vector space X, use the continuity of addition to show that there exists a neighborhood V of 0 such that $V + V \subseteq U$.

(d) If $X_1, \ldots, X_n$ are topological vector spaces, show that the (algebraic) direct sum $\bigoplus_{i=1}^n X_i$ is a topological vector space, with respect to the product topology. What about the direct product of infinitely many topological vector spaces?

(e) If Y is a linear subspace of X, show that Y is a topological vector space with respect to the relative topology.

(f) Show that, with respect to its Euclidean topology, $\mathbb{R}^n$ is a real topological vector space, and $\mathbb{C}^n$ is a complex topological vector space.

THEOREM 3.1. *Let X be a topological vector space. Then:*

1. *X is a regular topological space; i.e., if A is a closed subset of X and x is an element of X that is not in A, then there exist disjoint open sets U_1 and U_2 such that $x \in U_1$ and $A \subseteq U_2$.*
2. *X is connected.*
3. *X is compact if and only if X is $\{0\}$.*
4. *Every finite dimensional subspace Y of X is a closed subset of X.*

(5) *If T is a linear transformation of X into another topological vector space X', then T is continuous at each point of X if and only if T is continuous at the point $0 \in X$.*

PROOF. To see 1, let A be a closed subset of X and let x be a point of X not in A. Let U denote the open set $\tilde{A}$, and let U' be the open neighborhood $U - x$ of 0. (See part a of Exercise 3.1.) Let V be a neighborhood of 0 such that $V + V \subset U'$. Now $-V$ is a neighborhood of 0, and we let $W = V \cap (-V)$. Then $W = -W$ and $W + W \subset U'$. Let $U_1 = W + x$ and let $U_2 = W + A$. Then $x \in U_1$ and $A \subseteq U_2$. Clearly U_1 is an open set, and, because $U_2 = \cup_{y \in A}(W + y)$, we see also that U_2 is an open set. Further, if $z \in U_1 \cap U_2$, then we must have $z = x + w_1$ and $z = a + w_2$, where both w_1 and w_2 belong to W and $a \in A$. But then we would have

$$a = x + w_1 - w_2 \in x + W - W \subset x + U' = U = \tilde{A},$$

which is a contradiction. Therefore, $U_1 \cap U_2 = \emptyset$, and X is a regular topological space.

Because the map $t \to (1-t)x + ty$ is continuous on $\mathbb{R}$, it follows that any two elements of X can be joined by a curve, in fact by a line segment in X. Therefore, X is pathwise connected, hence connected, proving part 2.

Part 3 is left to an exercise.

We prove part 4 by induction on the dimension of the subspace Y. Although the assertion in part 4 seems simple enough, it is surprisingly difficult to prove. First, if Y has dimension 1, let $y \neq 0 \in Y$ be a basis for Y. If $\{t_\alpha y\}$ is a net in Y that converges to an element $x \in X$, then the net $\{t_\alpha\}$ must be eventually bounded in $\mathbb{R}$ (or $\mathbb{C}$), in the sense that there must exist an index α_0 and a constant M such that $|t_\alpha| \leq M$ for all $\alpha \geq \alpha_0$. Indeed, if the net $\{t_\alpha\}$ were not eventually bounded, let $\{t_{\alpha_\beta}\}$ be a subnet for which $\lim_\beta |t_{\alpha_\beta}| = \infty$. Then

$$\begin{aligned} y &= \lim_\beta (1/t_{\alpha_\beta}) t_{\alpha_\beta} y \\ &= \lim_\beta (1/t_{\alpha_\beta}) \lim_\beta t_{\alpha_\beta} y \\ &= 0 \times x \\ &= 0, \end{aligned}$$

which is a contradiction. So, the net $\{t_\alpha\}$ is bounded. Let $\{t_{\alpha_\beta}\}$ be a convergent subnet of $\{t_\alpha\}$ with limit t. Then

$$x = \lim_\alpha t_\alpha y = \lim_\beta t_{\alpha_\beta} y = ty,$$

whence $x \in Y$, and Y is closed.

Assume now that every $n-1$-dimensional subspace is closed, and let Y have dimension $n > 1$. Let $\{y_1, \ldots, y_n\}$ be a basis for Y, and write Y' for the linear span of $y_1, \ldots, y_{n-1}$. Then elements y of Y can be written uniquely in the form$y = y' + ty_n$, for $y' \in Y'$ and t real (complex). Suppose that x is an element of the closure of Y, i.e., $x = \lim_\alpha (y'_\alpha + t_\alpha y_n)$. As before, we have that the net $\{t_\alpha\}$ must be bounded. Indeed, if the net $\{t_\alpha\}$ were not bounded, then let $\{t_{\alpha_\beta}\}$ be a subnet for which $\lim_\beta |t_{\alpha_\beta}| = \infty$. Then

$$0 = \lim_\beta (1/t_{\alpha_\beta})x = \lim_\beta (y'_{\alpha_\beta}/t_{\alpha_\beta}) + y_n,$$

or

$$y_n = \lim_\beta -(y'_{\alpha_\beta}/t_{\alpha_\beta}),$$

implying that y_n belongs to the closure of the closed subspace Y'. Since y_n is linearly independent of the subspace Y', this is impossible, showing that the net $\{t_\alpha\}$ is bounded. Hence, letting $\{t_{\alpha_\beta}\}$ be a convergent subnet of $\{t_\alpha\}$, say $t = \lim_\beta t_{\alpha_\beta}$, we have

$$x = \lim_\beta (y'_{\alpha_\beta} + t_{\alpha_\beta} y_n),$$

showing that

$$x - ty_n = \lim y'_{\alpha_\beta},$$

whence, since Y' is closed, there exists a $y' \in Y'$ such that $x - ty_n = y'$. Therefore, $x = y' + ty_n \in Y$, and Y is closed, proving part 4.

Finally, if T is a linear transformation from X into X', then T being continuous at every point of X certainly implies that T is continuous at 0. Conversely, suppose T is continuous at 0, and let $x \in X$ be given. If V is a neighborhood of $T(x) \in X'$, let U be the neighborhood $V - T(x)$ of $0 \in X'$. Because T is continuous at 0, there exists a neighborhood W of $0 \in X$ such that $T(W) \subseteq U$. But then the neighborhood $W + x$ of x satisfies $T(W + x) \subseteq U + T(x) = V$, and this shows the continuity of T at x.

EXERCISE 3.2. (a) Prove part 3 of the preceding theorem.

(b) Prove that any linear transformation T, from $\mathbb{R}^n$ (or $\mathbb{C}^n$), equipped with its ordinary Euclidean topology, into a real (complex) topological vector space X, is necessarily continuous. HINT: Let $e_1, \ldots, e_n$ be the standard basis, and write $x_i = T(e_i)$.

(c) Let ρ be a seminorm (or subadditive functional) on a real topological vector space X. Show that ρ is continuous everywhere on X if and only if it is continuous at 0.

(d) Suppose ρ is a continuous seminorm on a real topological vector space X and that f is a linear functional on X that is bounded by ρ; i.e., $f(x) \leq \rho(x)$ for all $x \in X$. Prove that f is continuous.

(e) Suppose X is a vector space on which there is a topology $\mathcal{T}$ such that $(x, y) \to x - y$ is continuous from $X \times X$ into X. Show that $\mathcal{T}$ is Hausdorff if and only if it is T_0. (A topological space is called T_0 if, given any two points, there exists an open set that contains one of them but not the other.)

(f) Show that $L^p(\mathbb{R})$ is a topological vector space with respect to the topology defined by the (translation-invariant) metric

$$d(f, g) = \|f - g\|_p.$$

Show, in fact, that any normed linear space is a topological vector space with respect to the topology defined by the metric given by

$$d(x, y) = \|x - y\|.$$

(g) Let c_c denote the set of all real (or complex) sequences $\{a_1, a_2, \ldots\}$ that are nonzero for only finitely many terms. If $\{a_j\} \in c_c$, define the *norm* of $\{a_j\}$ by $\|\{a_j\}\| = \max_j |a_j|$. Verify that c_c is a normed linear space with respect to this definition of norm.

(h) Give an example of a (necessarily infinite dimensional) subspace of $L^p(\mathbb{R})$ which is not closed.

THEOREM 3.2. (Finite-Dimensional Topological Vector Spaces)

(1) *If X is a finite dimensional real (or complex) topological vector space, and if $x_1, \ldots, x_n$ is a basis for X, then the map $T : \mathbb{R}^n \to X$ (or $T : \mathbb{C}^n \to X$), defined by $T(t_1, \ldots, t_n) = \sum t_i x_i$, is a topological isomorphism of $\mathbb{R}^n$ (or $\mathbb{C}^n$), equipped with its Euclidean topology, onto X. That is, specifically, a net $\{x^\alpha\} = \{\sum_{i=1}^n t_i^\alpha x_i\}$ converges to an element $x = \sum_{i=1}^n t_i x_i \in X$ if and only if each net $\{t_i^\alpha\}$ converges to t_i, $1 \leq i \leq n$.*

(2) *The only topology on $\mathbb{R}^n$ (or $\mathbb{C}^n$), in which it is a topological vector space, is the usual Euclidean topology.*

(3) *Any linear transformation, from one finite dimensional topological vector space into another finite dimensional topological vector space, is necessarily continuous.*

PROOF. We verify these assertions for real vector spaces, leaving the complex case to the exercises. The map $T : \mathbb{R}^n \to X$ in part 1 is obviously linear, 1-1 and onto. Also, it is continuous by part b of Exercise 3.2. Let us show that T^{-1} is continuous. Thus, let the net $\{x^\alpha\} = \{\sum_{i=1}^n t_i^\alpha x_i\}$ converge to 0 in X. Suppose, by way of contradiction, that there exists an i for which the net $\{t_i^\alpha\}$ does not converge to 0. Then let $\{t_i^{\alpha^\beta}\}$ be a subnet for which $\lim_\beta t_i^{\alpha^\beta} = t$, where t either is $\pm\infty$ or is a nonzero real number. Write $x^\alpha = t_i^\alpha x_i + x'^\alpha$. Then

$$(1/t_i^{\alpha^\beta})x^{\alpha^\beta} = x_i + (1/t_i^{\alpha^\beta})x'^{\alpha^\beta},$$

whence,

$$x_i = -\lim_\beta (1/t_i^{\alpha^\beta})x'^{\alpha^\beta},$$

implying that x_i belongs to the (closed) subspace spanned by the vectors

$$x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n,$$

and this is a contradiction, since the x_i's form a basis of X. Therefore, each of the nets $\{t_i^\alpha\}$ converges to 0, and T^{-1} is continuous.

We leave the proofs of parts 2 and 3 to the exercises.

EXERCISE 3.3. (a) Prove parts 2 and 3 of the preceding theorem in the case that X is a real topological vector space.

(b) Prove the preceding theorem in the case that X is a complex topological vector space.

EXERCISE 3.4. (Quotient Topological Vector Spaces) Let M be a linear subspace of a topological vector space X.

(a) Prove that the natural map π, which sends $x \in X$ to $x + M \in X/M$, is continuous and is an open map, where X/M is given the quotient topology.

(b) Show that X/M, equipped with the quotient topology, is a topological vector space if and only if M is a closed subspace of X. HINT: Use part e of Exercise 3.2.

(c) Suppose M is not closed in X. Show that, if U is any neighborhood of $0 \in X$, then $U + M$ contains the closure $\overline{M}$ of M.

(d) Conclude from part c that, if M is dense in X, then the only open subsets of X/M are X/M and $\emptyset$.

THEOREM 3.3. *Let X be a real topological vector space. Then X is locally compact if and only if X is finite dimensional.*

PROOF. If X is finite dimensional it is clearly locally compact, since the only topology on $\mathbb{R}^n$ is the usual Euclidean one. Conversely, suppose U is a compact neighborhood of $0 \in X$, and let V be a neighborhood of 0 for which $V + V \subseteq U$. Because U is compact, there exists a finite set $x_1, \dots, x_n$ of points in U such that

$$U \subseteq \cup_{i=1}^{n}(x_i + V).$$

Let M denote the subspace of X spanned by the points $x_1, \dots, x_n$. Then M is a closed subspace, and the neighborhood $\pi(U)$ of 0 in X/M equals $\pi(V)$. Indeed, if $\pi(y) \in \pi(U)$, with $y \in U$, then there exists an $1 \leq i \leq n$ such that $y \in x_i + V$, whence $\pi(y) \in \pi(V)$.

It then follows that

$$\pi(U) = \pi(U) + \pi(U) = N\pi(U)$$

for every positive integer N,which implies that $\pi(U) = X/M$. So X/M is compact and hence is $\{0\}$. Therefore, $X = M$, and X is finite dimensional.

THEOREM 3.4. *Let T be a linear transformation of a real topological vector space X into a real topological vector space Y, and let M be the kernel of T. If π denotes the quotient map of X onto X/M, and if S is the unique linear transformation of the vector space X/M into Y satisfying $T = S \circ \pi$, then S is continuous if and only if T is continuous, and S is an open map if and only if T is an open map.*

PROOF. Since π is continuous and is an open map, see Exercise 3.4, It follows that T is continuous or open if S is continuous or open. If T is continuous, and if U is an open subset of Y, then $S^{-1}(U) = \pi(T^{-1}(U))$, and this is open because T is continuous and π is an open map. Hence, S is continuous.

Finally, if T is an open map and U is an open subset of X/M, then $S(U) = S(\pi(\pi^{-1}(U))) = T(\pi^{-1}(U))$, which is open because T is an open map and π is continuous. So, S is an open map.

THEOREM 3.5. (Characterization of Continuity) *If T is a linear transformation of a real (or complex) topological vector space X into $\mathbb{R}^n$ (or $\mathbb{C}^n$), then T is continuous if and only if $\ker(T)$ is closed. Further, T is continuous if and only if there exists a neighborhood of 0 in X*

on which T is bounded. If f is a linear functional on X, then f is continuous if and only if there exists a neighborhood of 0 on which f either is bounded above or is bounded below.

PROOF. Suppose that X is a real vector space. If $M = \ker(T)$ is closed, and if $T = S \circ \pi$, then T is continuous because S is, X/M being finite dimensional. The converse is obvious.

If T is not continuous, then, from the preceding paragraph, M is not closed. So, by part c of Exercise 3.4, every neighborhood U of 0 is such that $U + M$ contains $\overline{M}$. If x is an element of $\overline{M} - M$, then $T(x) \neq 0$. Also, for any scalar λ, $\lambda x \in \overline{M} \subseteq U + M$, whence there exists an $m \in M$ such that $\lambda x - m \in U$. But then, $T(\lambda x - m) = \lambda T(x)$, showing that T is not bounded on U. Again, the converse is immediate.

The third claim of this theorem follows in the same manner as the second, and the complex cases for all parts are completely analogous to the real ones.

REMARK. We shall see that the graph of a linear transformation is important vis a vis the continuity of T. The following exercise demonstrates the initial aspects of this connection.

EXERCISE 3.5. (Continuity and the Graph) Let X and Y be topological vector spaces, and let T be a linear transformation from X into Y.

(a) Show that if T is continuous then the graph of T is a closed subspace of $X \times Y$.

(b) Let X and Y both be the normed linear space c_c (see part g of Exercise 3.2), and define T by $T(\{a_j\}) = \{ja_j\}$. Verify that the graph of T is a closed subset of $X \times Y$ but that T is not continuous.

(c) Show that, if the graph of T is closed, then the kernel of T is closed.

(d) Let $Y = \mathbb{R}^n$ or $\mathbb{C}^n$. Show that T is continuous if and only if the graph of T is closed.

EXERCISE 3.6. (a) Let T be a linear transformation from a normed linear space X into a normed linear space Y. Show that T is continuous if and only if there exists a constant M such that

$$\|T(x)\| \leq M\|x\|$$

for every $x \in X$.

(b) Let X be an infinite dimensional normed linear space. Prove that there exists a discontinuous linear functional on X. HINT: Show that there exists an infinite set of linearly independent vectors of norm 1. Then, define a linear functional that is not bounded on any neighborhood of 0.

(c) Show that, if $1 \leq p < \infty$, then $L^p(\mathbb{R})$ is a separable normed linear space. What about $L^\infty(\mathbb{R})$?

(d) Let μ be counting measure on an uncountable set X. Show that each $L^p(\mu)$ $(1 \leq p \leq \infty)$ is a normed linear space but that none is separable.

(e) Let Δ be a second-countable, locally compact, Hausdorff, topological space. Show that $X = C_0(\Delta)$ is a separable normed linear space, where the norm on X is the supremum norm. (See Exercise 1.9.)

DEFINITION. Let X be a set, and let $\{f_\nu\}$ be a collection of real-valued (or complex-valued) functions on X. The *weak topology* on X, *generated* by the f_ν's, is the smallest topology on X for which each f_ν is continuous. A basis for this topology consists of sets of the form

$$V = \cap_{i=1}^n f_{\nu_i}^{-1}(U_i),$$

where each U_i is an open subset of $\mathbb{R}$ (or $\mathbb{C}$).

EXERCISE 3.7. (Vector Space Topology Generated by a Set of Linear Functionals) Let X be a real vector space and let $\{f_\nu\}$ be a collection of linear functionals on X that separates the points of X. Let $Y = \prod_\nu \mathbb{R}$, and define a function $F : X \to Y$ by $[F(x)](\nu) = f_\nu(x)$.

(a) Show that F is 1-1, and that with respect to the weak topology on X, generated by the f_ν's, F is a homeomorphism of X onto the subset $F(X)$ of Y. HINT: Compare the bases for the two topologies.

(b) Conclude that convergence in the weak topology on X, generated by the f_ν's, is described as follows:

$$x = \lim_\alpha x_\alpha \ \equiv \ f_\nu(x) = \lim_\alpha f_\nu(x_\alpha)$$

for all ν.

(c) Prove that X, equipped with the weak topology generated by the f_ν's, is a topological vector space.

(d) Show that Y is metrizable, and hence this weak topology on X is metrizable, if the set of f_ν's is countable.

(e) Verify that parts a through d hold if X is a complex vector space and each f_ν is a complex linear functional.

An important kind of topological vector space is obtained as a generalization of the preceding exercise, and is constructed as follows. Let X

be a (real or complex) vector space, and let $\{\rho_\nu\}$ be a collection of seminorms on X that separates the nonzero points of X from 0 in the sense that for each $x \neq 0$ there exists a ν such that $\rho_\nu(x) > 0$. For each $y \in X$ and each index ν, define $g_{y,\nu}(x) = \rho_\nu(x - y)$. Then X, equipped with the weakest topology making all of the $g_{y,\nu}$'s continuous, is a topological vector space, i.e., is Hausdorff and addition and scalar multiplication are continuous. A net $\{x_\alpha\}$ of elements in X converges in this topology to an element x if and only if $\rho_\nu(x - x_\alpha)$ converges to 0 for every ν. Further, this topology is a metrizable topology if the collection $\{\rho_\nu\}$ is countable.

We call this the *vector space topology* on X *generated* by the seminorms $\{\rho_\nu\}$ and denote this topological vector space by $(X, \{\rho_\nu\})$.

If $\rho_1, \rho_2, \ldots$ is a sequence of norms on X, then we call the topological vector space $(X, \{\rho_n\})$ a *countably normed* space.

EXERCISE 3.8. (Vector Space Topology Generated by a Set of Seminorms) Let X be a real (or complex) vector space and let $\{\rho_\nu\}$ be a collection of seminorms on X that separates the nonzero points of X from 0 in the sense that for each $x \neq 0$ there exists a ν such that $\rho_\nu(x) > 0$. For each $y \in X$ and each index ν, define $g_{y,\nu}(x) = \rho_\nu(x - y)$. Finally, let $\mathcal{T}$ be the topology on X generated by the $g_{y,\nu}$'s.

(a) Let x be an element of X and let V be an open set containing x. Show that there exist indices $\nu_1, \ldots, \nu_n$, elements $y_1, \ldots, y_n \in X$, and open sets $U_1, \ldots, U_n \subseteq \mathbb{R}$ $(\mathbb{C})$ such that

$$x \in \cap_{i=1}^n g_{y_i,\nu_i}^{-1}(U_i) \subseteq V.$$

(b) Conclude that convergence in the topology on X generated by the $g_{y,\nu}$'s is described by

$$x = \lim_\alpha x_\alpha \equiv \lim_\alpha \rho_\nu(x - x_\alpha) = 0$$

for each ν.

(c) Prove that X, equipped with the topology generated by the $g_{y,\nu}$'s, is a topological vector space. (HINT: Use nets.) Show further that this topology is metrizable if the collection $\{\rho_\nu\}$ is countable, i.e., if $\rho_1, \rho_2, \ldots$ is a sequence of seminorms. (HINT: Use the formula

$$d(x, y) = \sum_{n=1}^\infty 2^{-n} \min(\rho_n(x - y), 1).$$

Verify that d is a translation-invariant metric and that convergence with respect to this metric is equivalent to convergence in the topology $\mathcal{T}$.)

(d) Let X be a vector space, and let $\rho_1, \rho_2, \dots$ be a sequence of seminorms that separate the nonzero points of X from 0. For each $n \geq 1$, define $p_n = \max_{k \leq n} \rho_k$. Prove that each p_n is a seminorm on X, that $p_n \leq p_{n+1}$ for all n, and that the two topological vector spaces $(X, \{\rho_n\})$ and $(X, \{p_n\})$ are topologically isomorphic.

(e) Let X and $\{p_n\}$ be as in part d. Show that if V is a neighborhood of 0, then there exists an integer n and an $\epsilon > 0$ such that if $p_n(x) < \epsilon$, then $x \in V$. Deduce that, if f is a continuous linear functional on $(X, \{p_n\})$, then there exists an integer n and a constant M such that $|f(x)| \leq M p_n(x)$ for all $x \in X$.

(f) Let X be a normed linear space, and define $\rho(x) = \|x\|$. Prove that the topology on X determined by the norm coincides with the vector space topology generated by ρ.

EXERCISE 3.9. (a) Let X be the complex vector space of all infinitely differentiable complex-valued functions on $\mathbb{R}$. For each nonnegative integer n, define ρ_n on X by

$$\rho_n(f) = \sup_{|x| \leq n} \sup_{0 \leq i \leq n} |f^{(i)}(x)|,$$

where $f^{(i)}$ denotes the ith derivative of f. Show that the ρ_n's are seminorms (but not norms) that separate the nonzero points of X from 0, whence X is a metrizable complex topological vector space in the weak vector space topology generated by the ρ_n's. This vector space is usually denoted by $\mathcal{E}$.

(b) Let X be the complex vector space $C_0(\Delta)$, where Δ is a locally compact Hausdorff space. For each $\delta \in \Delta$, define ρ_δ on X by

$$\rho_\delta(f) = |f(\delta)|.$$

Show that, with respect to the weak vector space topology generated by the ρ_δ's, convergence is pointwise convergence of the functions.

EXERCISE 3.10. (Schwartz Space) Let $\mathcal{S}$ denote the set of all C^∞ complex-valued functions f on $\mathbb{R}$ that are *rapidly decreasing,* i.e., such that $x^n f^{(j)}(x) \in C_0(\mathbb{R})$ for every pair of nonnegative integers n and j. In other words, f and all its derivatives tend to 0 at $\pm\infty$ faster than the reciprocal of any polynomial.

(a) Show that every C^∞ function having compact support belongs to $\mathcal{S}$, and verify that $f(x) = x^k e^{-x^2}$ belongs to $\mathcal{S}$ for every integer $k \geq 0$.

(b) Show that $\mathcal{S}$ is a complex vector space, that each element of $\mathcal{S}$ belongs to every L^p space, and that $\mathcal{S}$ is closed under differentiation and multiplication by polynomials. What about antiderivatives of elements of $\mathcal{S}$? Are they again in $\mathcal{S}$?

(c) For each nonnegative integer n, define p_n on $\mathcal{S}$ by

$$p_n(f) = \max_{0\leq i,j\leq n} \sup_x |x^j f^{(i)}(x)|.$$

Show that each p_n is a norm on $\mathcal{S}$, that $p_n(f) \leq p_{n+1}(f)$ for all $f \in \mathcal{S}$, and that the topological vector space $(\mathcal{S}, \{p_n\})$ is a countably normed space. This countably normed vector space is called *Schwartz space*.

(d) Show that $f = \lim f_k$ in $\mathcal{S}$ if and only if $\{x^j f_k^{(i)}(x)\}$ converges uniformly to $x^j f^{(i)}(x)$ for every i and j.

(e) Prove that the set $\mathcal{D}$ of C^∞ functions having compact support is a dense subspace of $\mathcal{S}$. HINT: Let χ be a nonnegative C^∞ function, supported on $[-2, 2]$, and satisfying $\chi(x) = 1$ for $-1 \leq x \leq 1$. Define $\chi_n(x) = \chi(x/n)$. If $f \in \mathcal{S}$, show that $f = \lim f\chi_n$ in the topology of $\mathcal{S}$.

(f) Prove that the map $f \to f'$ is a continuous linear transformation from $\mathcal{S}$ into itself. Is this transformation onto?

We introduce next a concept that is apparently purely from algebraic linear space theory and one that is of extreme importance in the topological aspect of Functional Analysis.

DEFINITION. A subset S of a vector space is called *convex* if $(1-t)x + ty \in S$ whenever $x, y \in S$ and $0 \leq t \leq 1$. The *convex hull* of a set S is the smallest convex set containing S (the intersection of all convex sets containing S). A topological vector space X is called *locally convex* if there exists a neighborhood basis at 0 consisting of convex subsets of X. That is, if U is any neighborhood of 0 in X, then there exists a convex open set V such that $0 \in V \subseteq U$.

EXERCISE 3.11. (a) Let X be a real vector space. Show that the intersection of two convex subsets of X and the sum of two convex subsets of X is a convex set. If S is a subset of X, show that the intersection of all convex sets containing S is a convex set. Show also that if X is a topological vector space then the closure of a convex set is convex.

(b) Prove that a normed linear space is locally convex by showing that each ball centered at 0 in X is a convex set.

(c) Let X be a vector space and let $\{f_\nu\}$ be a collection of linear functionals on X that separates the points of X. Show that X, equipped with the weakest topology making all of the f_ν's continuous, is a locally convex topological vector space. (See Exercise 3.7.)

(d) Let X be a vector space, and let $\{\rho_\nu\}$ be a collection of seminorms on X that separates the nonzero points of X from 0. Prove that X, equipped with the weak vector space topology generated by the ρ_ν's, is a locally convex topological vector space. (See Exercise 3.8.)

(e) Suppose X is a locally convex topological vector space and that M is a subspace of X. Show that M is a locally convex topological vector space with respect to the relative topology. If M is a closed subspace of X, show that the quotient space X/M is a locally convex topological vector space.

(f) Show that all the L^p spaces are locally convex as well as the spaces $C_0(\Delta)$ under pointwise convergence, $\mathcal{E}$, and $\mathcal{S}$ of Exercises 3.9 and 3.10.

If X is a real vector space, recall that a function $\rho : X \to \mathbb{R}$ is called a *subadditive functional* if

(1) $\rho(x+y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$.
(2) $\rho(tx) = t\rho(x)$ for all $x \in X$ and $t \geq 0$.

THEOREM 3.6. (Convex Neighborhoods of 0 and Continuous Subadditive Functionals) *Let X be a real topological vector space. If ρ is a continuous subadditive functional on$X$, then $\rho^{-1}(-\infty, 1)$ is a convex neighborhood of 0 in X. Conversely, if U is a convex neighborhood of 0, then there exists a continuous nonnegative subadditive functional ρ such that $\rho^{-1}(-\infty, 1) \subseteq U \subseteq \rho^{-1}(-\infty, 1]$. In addition, if U is symmetric, then ρ may be chosen to be a seminorm.*

PROOF. If ρ is a continuous subadditive functional, then it is immediate that $\rho^{-1}(-\infty, 1)$ is open, contains 0, and is convex.

Conversely, if U is a convex neighborhood of 0, define ρ on X by

$$\rho(x) = \frac{1}{\sup_{t>0, tx \in U} t} = \inf_{r>0, x \in rU} r.$$

(We interpret $\rho(x)$ as 0 if the supremum in the denominator is ∞, i.e., if $x \in rU$ for all $r > 0$.) Because U is an open neighborhood of $0 = 0 \times x$, and because scalar multiplication is continuous, the supremum in the above formula is always > 0, so that $0 \leq \rho(x) < \infty$ for every x. Notice also that if $t > 0$ and $t \times x \in U$, then $1/t \geq \rho(x)$.

It follows immediately that $\rho(rx) = r\rho(x)$ if $r \geq 0$, and, if U is symmetric, then $\rho(rx) = |r|\rho(x)$ for arbitrary real r.

If x and y are in X and $\epsilon > 0$ is given, choose real numbers t and s such that $tx \in U$, $sy \in U$, $1/t \leq \rho(x) + \epsilon$, and $1/s \leq \rho(y) + \epsilon$. Because U

is convex, we have that

$$\frac{s}{t+s}tx \ + \ \frac{t}{t+s}sy = \frac{st}{t+s}(x+y) \in U.$$

Therefore, $\rho(x+y) \leq (s+t)/st$, whence

$$\rho(x+y) \leq (t+s)/st = (1/t) + (1/s) \leq \rho(x) + \rho(y) + 2\epsilon,$$

completing the proof that ρ is a subadditive functional in general and a seminorm if U is symmetric.

If $\rho(x) < 1$, then there exists a $t > 1$ so that $tx \in U$. Since U is convex, it then follows that $x \in U$. Also, if $x = 1 \times x \in U$, then $\rho(x) \leq 1$. Hence, $\rho^{-1}(-\infty, 1) \subseteq U \subseteq \rho^{-1}(-\infty, 1]$.

Finally, $\rho^{-1}(-\infty, \epsilon) \subseteq \epsilon U \subseteq \rho^{-1}(-\infty, \epsilon]$ for every positive ϵ, which shows that ρ is continuous at 0 and hence everywhere.

REMARK. The subadditive functional ρ constructed in the preceding proof is called the *Minkowski functional* associated to the convex neighborhood U.

THEOREM 3.7. (Hahn-Banach Theorem, Locally Convex Version) *Let X be a real locally convex topological vector space, let Y be a subspace of X, and let f be a continuous linear functional on Y with respect to the relative topology. Then there exists a continuous linear functional g on X whose restriction to Y is f.*

PROOF. By Theorem 3.5, there exists a neighborhood V of 0 in Y on which f is bounded, and by scaling we may assume that it is bounded by 1; i.e., $|f(y)| \leq 1$ if $y \in V$. Let W be a neighborhood of 0 in X such that $V = W \cap Y$, and let U be a symmetric convex neighborhood of 0 in X such that $U \subseteq W$. Let ρ be the continuous seminorm (Minkowski functional) on X associated to U as in the preceding theorem.

Now, if $y \in Y$, $t > 0$, and $ty \in U$, then

$$|f(y)| = (1/t)|f(ty)| \leq 1/t,$$

whence, by taking the supremum over all such t's,

$$|f(y)| \leq \rho(y),$$

showing that f is bounded by ρ on Y. Using Theorem 2.2, let g be a linear functional on X that extends f and such that $|g(x)| \leq \rho(x)$ for all $x \in X$. Then g is an extension of f and is continuous, so the proof is complete.

EXERCISE 3.12. Let M be a subspace of a locally convex topological vector space X. Prove that M is dense in X if and only if the only continuous linear functional f on X that is identically 0 on M is the 0 functional.

THEOREM 3.8. (Local Convexity and Existence of Continuous Linear Functionals) *A locally convex topological vector space has sufficiently many continuous linear functionals to separate its points.*

PROOF. Assume first that X is a real topological vector space. We will apply the Hahn-Banach Theorem. Suppose that $x \neq y$ are elements of X, and let Y be the subspace of X consisting of the real multiples of the nonzero vector $y - x$. Define a linear functional f on Y by

$$f(t(y-x)) = t.$$

Because Y is one-dimensional, this linear functional f is continuous. By the Hahn-Banach Theorem above, there exists a continuous linear functional g on X that is an extension of f. We have that

$$g(y) - g(x) = g(y-x) = f(y-x) = 1 \neq 0,$$

showing that g separates the two points x and y.

Now, if X is a complex locally convex topological vector space, then it is obviously a real locally convex topological vector space. Hence, if $x \neq y$ are elements of X, then there exists a continuous real linear functional g on X such that $g(x) \neq g(y)$. But, as we have seen in Chapter I, the formula

$$f(z) = g(z) - ig(iz)$$

defines a complex linear functional on X, and clearly f is continuous and $f(x) \neq f(y)$.

EXERCISE 3.13. (Example of a Non-Locally-Convex Topological Vector Space) Let X' be the vector space of all real-valued Lebesgue measurable functions on $[0, 1]$. and define an equivalence relation $\equiv$ on X' by $f \equiv g$ if $f(x) = g(x)$ a.e.m, where m denotes Lebesgue measure. For $f, g \in X'$, set

$$d'(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} m(\{x : |f(x) - g(x)| \geq \frac{1}{n}\}).$$

(a) Prove that $X = X'/\equiv$ is a vector space, and show that d' determines a translation-invariant metric d on X. HINT: Show that $\{x : |f(x) - h(x)| \geq 1/n\}$ is a subset of $\{x : |f(x) - g(x)| \geq 1/2n\} \cup \{x : |g(x) - h(x)| \geq 1/2n\}$.

(b) Show that $d'(f_n - f) \to 0$ if and only if the sequence $\{f_n\}$ converges in measure to f. Conclude that the metric d is a complete metric.

(c) Prove that, with respect to the topology on X determined by the metric d, X is a topological vector space (in fact a Fréchet space), and that the subspace $Y \subseteq X$ consisting of the equivalence classes $[\phi]$ corresponding to measurable simple functions ϕ is dense in X.

(d) Let $\delta > 0$ be given. Show that if E is a measurable set of measure $< \delta$, then for every scalar $c \geq 1$ the equivalence class $[c\chi_E]$ belongs to the ball $B_\delta(0)$ of radius δ around $0 \in X$.

(e) Let f be a continuous linear functional on the topological vector space X, and let $B_\delta(0)$ be a neighborhood of $0 \in X$ on which f is bounded. See Theorem 3.5. Show that $f([\chi_E]) = 0$ for all E with $m(E) < \delta$, whence $f([\phi]) = 0$ for all measurable simple functions ϕ.

(f) Conclude that the only continuous linear functional on X is the zero functional, whence the topology on X is not locally convex.

THEOREM 3.9. (Separation Theorem) *Let C be a closed convex subset of a locally convex real topological vector space X, and let x be an element of X that is not in C. Then there exists a continuous linear functional ϕ on X and a real number s such that $\phi(c) \leq s < \phi(x)$ for all $c \in C$.*

PROOF. Again, we apply the Hahn-Banach Theorem. Let U be a neighborhood of 0 such that $x + U$ does not intersect C. Let V be a convex symmetric neighborhood of 0 for which $V + V \subseteq U$, and write C' for the open convex set $V + C$. Then $(x + V) \cap C' = \emptyset$. If y is an element of C', write W for the convex neighborhood $C' - y$ of 0, and observe that $(x - y + V) \cap W = \emptyset$. Let ρ be the continuous subadditive functional associated to W as in Theorem 3.6. (ρ is not necessarily a seminorm since W need not be symmetric.) If Y is the linear span of the nonzero vector $x - y$, let f be defined on Y by $f(t(x - y)) = t\rho(x - y)$. Then f is a linear functional on Y satisfying $f(z) \leq \rho(z)$ for all $z \in Y$. By part c of Exercise 2.6, there exists a linear functional ϕ on X, which is an extension of f and which satisfies $\phi(w) \leq \rho(w)$ for all $w \in X$.

Since ρ is continuous, it follows that ϕ is continuous. Also, by the definition of ρ, if $z \in W$, then $\rho(z) \leq 1$, whence $\phi(z) \leq 1$. Now $\rho(x-y) > 1$. For, if t is sufficiently close to 1, then $t(x - y) \in x - y + V$, whence

$t(x-y) \notin W$, and $\rho(x-y) \geq 1/t > 1$. So, $\phi(x-y) = f(x-y) = \rho(x-y) > 1$. Setting $s = \phi(y)+1$, we have $\phi(c) \leq s$ for all $c \in C$, and $\phi(x) > s$, as desired.

DEFINITION. Let C be a convex subset of a real vector space X. We say that a nonempty convex subset F of C is a *face* of C if: Whenever $x \in F$ is a *proper convex combination* of points in C (i.e., $x = (1-t)y + tz$, with $y \in C,\ z \in C$, and $0 < t < 1$,) then both y and z belong to F.

A point $x \in C$ is called an *extreme point* of C if: Whenever $x = (1-t)y + tz$, with $y \in C,\ z \in C$, and $0 < t < 1$, then $y = z = x$.

EXERCISE 3.14. (a) Let C be the closed unit ball in $L^p(\mathbb{R})$, for $1 < p < \infty$. Show that the extreme points of C are precisely the elements of the unit sphere, i.e., the elements f for which $\|f\|_p = 1$. HINT: Use the fact that $|(1-t)y + tz|^p < (1-t)|y|^p + t|z|^p$ if $y \neq z$ and $0 < t < 1$.

(b) If C is the closed unit ball in $L^1(\mathbb{R})$, show that C has no extreme points.

(c) Find the extreme points of the closed unit ball in $l^\infty(\mathbb{R})$.

(d) Find all the faces of a right circular cylinder, a tetrahedron, a sphere. Are all these faces closed sets?

(e) Suppose C is a closed convex subset of a topological vector space X. Is the closure of a face of C again a face? Is every face of C necessarily closed?

(f) Show that a singleton, which is a face of a convex set C, is an extreme point of C.

(g) Suppose C is a convex subset of a topological vector space X. Show that the intersection of two faces of C is a face of C. Also, if ϕ is a linear functional on X, and $\max_{x \in C} \phi(x) = c$, show that $\phi^{-1}(c) \cap C$ is a face of C.

EXERCISE 3.15. (Hahn-Banach Theorem, Extreme Point Version) Let X be a real vector space, and let ρ be a seminorm (or subadditive functional) on X. If Z is a subspace of X, define F_Z to be the set of all linear functionals f on Z for which $f(z) \leq \rho(z)$ for all $z \in Z$.

(a) Prove that F_Z is a convex set of linear functionals.

(b) Let Y be a subspace of X. If f is an extreme point of F_Y, show that there is an extreme point $g \in F_X$ that is an extension of f. HINT: Mimic the proof of Theorem 2.2. That is, use the Hausdorff maximality principle to find a maximal pair (Z, h), for which h is an extension of f and h is an extreme point of F_Z. Then, following the notation in the proof to Theorem 2.2, show that $Z = X$ by choosing c to equal b.

We give two main theorems concerning the set of extreme points of a convex set.

THEOREM 3.10. (Krein-Milman Theorem) *Let C be a nonempty compact convex subset of a locally convex real topological vector space X. Then*

(1) *There exists an extreme point of C.*
(2) *C is the closure of the convex hull of its extreme points.*

PROOF. Let $\mathcal{F}$ be the collection of all closed faces of C, and consider $\mathcal{F}$ to be a partially ordered set by defining $F \leq F'$ if $F' \subseteq F$. Then, $\mathcal{F}$ is nonempty (C is an element of $\mathcal{F}$), and we let $\{F_\alpha\}$ be a maximal linearly ordered subset of $\mathcal{F}$ (the Hausdorff maximality principle). We set $F = \cap F_\alpha$, and note, since C is compact, that F is a nonempty closed (compact) face of C. We claim that F is a singleton, whence an extreme point of C. Indeed, if $x \in F,\ y \in F$, and $x \neq y$, let ϕ be a continuous linear functional which separates x and y, and let z be a point in the compact set F at which ϕ attains its maximum on F. Let $H = \phi^{-1}(\phi(z))$, and let $F' = F \cap H$. Then F' is a closed face of C which is properly contained in F. See part g of Exercise 3.14. But then the subset of $\mathcal{F}$, consisting of the F_α's together with F', is a strictly larger linearly ordered subset of $\mathcal{F}$, and this is a contradiction. Therefore, F is a singleton, and part 1 is proved.

Next, let C' be the closure of the convex hull of the extreme points of C. Then $C' \subseteq C$. If there is an $x \in C$ which is not in C', then, using the Separation Theorem (Theorem 3.9), let s be a real number and ϕ be a continuous linear functional for which $\phi(y) \leq s < \phi(x)$ for all $y \in C'$. Because C is compact and ϕ is continuous, there exists a $z \in C$ such that $\phi(z) \geq \phi(w)$ for all $w \in C$, and we let $C'' = C \cap \phi^{-1}(\phi(z))$. Then C'' is a nonempty compact convex subset of C, and $C' \cap C'' = \emptyset$. By part 1, there exists an extreme point p of C''. We claim that p is also an extreme point of C. Thus, if $p = (1-t)q + tr$, with $q \in C,\ r \in C$, and $0 < t < 1$, then

$$\begin{aligned}\phi(z) &= \phi(p)\\ &= (1-t)\phi(q) + t\phi(r)\\ &\leq (1-t)\phi(z) + t\phi(z)\\ &= \phi(z).\end{aligned}$$

Therefore, $\phi(q) = \phi(r) = \phi(z)$, which implies that $q \in C''$ and $r \in C''$. Then, since p is an extreme point of C'', we have that $q = r = p$, as desired. But this implies that $p \in C'$, which is a contradiction. This completes the proof of part 2.

The Krein-Milman theorem is a topological statement about the set of extreme points of a compact convex set. Choquet's theorem, to follow, is a measure-theoretic statement about the set of extreme points of a compact convex set.

THEOREM 3.11. (Choquet Theorem) *Let X be a locally convex real topological vector space, let K be a metrizable, compact, convex subset of X, and let E denote the set of extreme points of K. Then:*

(1) *E is a Borel subset of K.*
(2) *For each $x \in K$, there exists a Borel probability measure μ_x on E such that*
$$f(x) = \int_E f(q)\, d\mu_x(q),$$
for every continuous linear functional f on X.

PROOF. Let A be the complement in $K \times K$ of the diagonal, i.e., the complement of the set of all pairs (x, x) for $x \in K$. Then A is an open subset of a compact metric space, and therefore A is a countable increasing union $A = \cup A_n$ of compact sets $\{A_n\}$. Define a function $I : (0, 1) \times A \to K$ by $I(t, y, z) = (1 - t)y + tz$. Then the range of I is precisely the complement of E in K. Also, since I is continuous, the range of I is the countable union of the compact sets $I([1/n, 1 - 1/n] \times A_n)$, whence the complement of E is an F_σ subset of K, so that E is a G_δ, hence a Borel set. This proves part 1.

Now, let Y denote the vector space of all continuous affine functions on K, i.e., all those continuous real-valued functions g on K for which

$$g((1 - t)y + tz) = (1 - t)g(y) + tg(z)$$

for all $y, z \in K$ and $0 \leq t \leq 1$. Note that the restriction to K of any continuous linear functional on X is an element of Y. Now Y is a subspace of $C(K)$. Since K is compact and metrizable, we have that $C(K)$ is a separable normed linear space in the uniform norm, whence Y is a separable normed linear space. Let $\{g_1, g_2, \dots\}$ be a countable dense set in the unit ball $B_1(0)$ of Y, and define

$$g' = \sum_{i=1}^{\infty} 2^{-i} g_i^2.$$

Then g' is continuous on K, and is a proper convex function; i.e.,

$$g'((1-t)y+tz) < (1-t)g'(y)+tg'(z)$$

whenever $y, z \in K$, $y \neq z$, and $0 < t < 1$. Indeed, the series defining g' converges uniformly by the Weierstrass M test, showing that g' is continuous. Also, if $y, z \in K$, with $y \neq z$, there exists a continuous linear functional ϕ on X that separates y and z. In fact, any nonzero multiple of ϕ separates y and z. So, there exists at least one i such that $g_i(y) \neq g_i(z)$. Now, for any such i, if $0 < t < 1$, then

$$g_i^2((1-t)y+tz) < (1-t)g_i^2(y)+tg_i^2(z),$$

since

$$((1-t)a+tb)^2-(1-t)a^2-tb^2 < 0$$

for all $a \neq b$. Indeed, this function of b is 0 when $b = a$ and has a negative derivative for $b > a$. On the other hand, if i is such that $g_i(y) = g_i(z)$, then

$$g_i^2((1-t)y+tz) = (g_i((1-t)y+tz))^2 = g_i^2(y) = (1-t)g_i^2(y)+tg_i^2(z).$$

Hence,

$$\begin{aligned}
g'((1-t)y+tz) &= \sum_{i=1}^{\infty} 2^{-i} g_i^2((1-t)y+tz) \\
&< \sum_{i=1}^{\infty} 2^{-i}[(1-t)g_i^2(y)+tg_i^2(z)] \\
&= (1-t)g'(y)+tg'(z).
\end{aligned}$$

We let Y_1 be the linear span of Y and g', so that we may write each element of Y_1 as $g + rg'$, where $g \in Y$ and $r \in \mathbb{R}$.

Now, given an $x \in K$, define a function ρ_x on $C(K)$ by

$$\rho_x(h) = \inf c(x),$$

where the infimum is taken over all continuous concave functions c on K for which $h(y) \leq c(y)$ for all $y \in K$. Recall that a function c on K is called *concave* if

$$c((1-t)y+tz) \geq (1-t)c(y)+tc(z),$$

for all $y, z \in K$ and $0 \leq t \leq 1$. Because the sum of two concave functions is again concave and a positive multiple of a concave function is again concave, it follows directly that ρ_x is a subadditive functional on $C(K)$. Note also that if c is a continuous concave function on K, then $\rho_x(c) = c(x)$. Define a linear functional ψ_x on Y_1 by

$$\psi_x(g + rg') = g(x) + r\rho_x(g').$$

Note that the identically 1 function I is an affine function, so it belongs to Y and hence to Y_1. It follows then that $\psi_x(I) = 1$. Also, we have that $\psi_x \leq \rho_x$ on Y_1 (see the exercise following), and we let ϕ_x be a linear functional on $C(K)$, which is an extension of ψ_x, and for which $\phi_x \leq \rho_x$ on $C(K)$. (We are using part c of Exercise 2.6.)

Note that, if $h \in C(K) \leq 0$, then $\rho_x(h) \leq 0$ (the 0 function is concave and $0 \geq h$). So, if $h \leq 0$, then $\phi_x(h) \leq \rho_x(h) \leq 0$. It follows from this that ϕ_x is a positive linear functional. By the Riesz Representation Theorem, we let ν_x be the unique (finite) Borel measure on K for which

$$\phi_x(h) = \int h \, d\nu_x$$

for all $h \in C(K)$. Again letting I denote the identically 1 function on K, we have that

$$\begin{aligned} \nu_x(K) &= \int I \, d\nu_x \\ &= \phi_x(I) \\ &= \psi_x(I) \\ &= 1, \end{aligned}$$

showing that ν_x is a probability measure.

If f is a continuous linear functional on X, then

$$\int f \, d\nu_x = \phi_x(f) = \psi_x(f) = f(x),$$

since the restriction of f to K is a continuous affine function, whence in Y_1.

We prove next that ν_x is supported on E. To do this, let $\{c_n\}$ be a sequence of continuous concave functions on K for which $c_n \geq g'$ for all n and $\rho_x(g') = \lim c_n(x)$. Set $c = \liminf c_n$. Then c is a Borel function, hence is ν_x-measurable, and $c(y) \geq g'(y)$ for all $y \in K$. Hence, $\int (c - g')\, d\nu_x \geq 0$. But,

$$\begin{aligned}
\int (c - g')\, d\nu_x &= \int (\liminf c_n - g')\, d\nu_x \\
&\leq \liminf \int (c_n - g')\, d\nu_x \\
&= \liminf \phi_x(c_n - g') \\
&= \liminf \phi_x(c_n) - \phi_x(g') \\
&= \liminf \phi_x(c_n) - \rho_x(g') \\
&\leq \liminf \rho_x(c_n) - \rho_x(g') \\
&= \liminf c_n(x) - \rho_x(g') \\
&= \lim c_n(x) - \rho_x(g') \\
&= 0.
\end{aligned}$$

Therefore, ν_x is supported on the set where c and g' agree. Let us show that $c(w) \neq g'(w)$ whenever $w \notin E$. Thus, if $w = (1-t)y + tz$, for $y, z \in K,\ y \neq z$, and $0 < t < 1$, then

$$\begin{aligned}
c(w) &= \liminf c_n(w) \\
&= \liminf c_n((1-t)y + tz) \\
&\geq \liminf[(1-t)c_n(y) + tc_n(z)] \\
&\geq (1-t)g'(y) + tg'(z) \\
&> g'((1-t)y + tz) \\
&= g'(w).
\end{aligned}$$

Define μ_x to be the restriction of ν_x to E. Then μ_x is a Borel probability measure on E, and

$$\int_E f\, d\mu_x = \int_K f\, d\nu_x = f(x)$$

for all continuous linear functionals f on X. This completes the proof.

EXERCISE 3.16. (a) Verify that the function ρ_x in the preceding proof is a subadditive functional and that $\psi_x(h) \leq \rho_x(h)$ for all $h \in Y_1$.

(b) Let $X = \mathbb{R}^2$, let $K = \{(s,t) : |s| + |t| \leq 1\}$, and let $x = (0,0)$ be the origin. Show that there are uncountably many different Borel probability measures μ on the set E of extreme points of K for which $f(x) = \int_E f(q)\, d\mu(q)$ for all linear functionals on X. Conclude that there can be no uniqueness assertion in Choquet's Theorem.