CHAPTER V

DUAL SPACES

DEFINITION Let $(X, \mathcal{T})$ be a (real) locally convex topological vector space. By the *dual space* X^*, or $(X, \mathcal{T})^*$, of X we mean the set of all continuous linear functionals on X.

By the *weak topology* on X we mean the weakest topology $\mathcal{W}$ on X for which each $f \in X^*$ is continuous. In this context, the topology $\mathcal{T}$ is called the *strong topology* or *original topology* on X.

EXERCISE 5.1. (a) Prove that X^* is a vector space under pointwise operations.

(b) Show that $\mathcal{W} \subseteq \mathcal{T}$. Show also that $(X, \mathcal{W})$ is a locally convex topological vector space.

(c) Show that if X is infinite dimensional then every weak neighborhood of 0 contains a nontrivial subspace M of X. HINT: If $V = \cap_{i=1}^n f_i^{-1}(U_i)$, and if $M = \cap_{i=1}^n \ker(f_i)$, then $M \subseteq V$.

(d) Show that a linear functional f on X is strongly continuous if and only if it is weakly continuous; i.e., prove that $(X, \mathcal{T})^* = (X, \mathcal{W})^*$.

(e) Prove that X is finite dimensional if and only if X^* is finite dimensional, in which case X and X^* have the same dimension.

EXERCISE 5.2. (a) For $1 < p < \infty$, let X be the normed linear space $L^p(\mathbb{R})$. For each $g \in L^{p'}(\mathbb{R})$ $(1/p + 1/p' = 1)$, define a linear functional ϕ_g on X by

$$\phi_g(f) = \int f(x)g(x)\,dx.$$

Prove that the map $g \to \phi_g$ is a vector space isomorphism of $L^{p'}(\mathbb{R})$ onto X^*.

(b) By analogy to part a, show that $L^\infty(\mathbb{R})$ is isomorphic as a vector space to $L^1(\mathbb{R})^*$.

(c) Let c_0 be the normed linear space of real sequences $\{a_0, a_1, \dots\}$ for which $\lim a_n = 0$ with respect to the norm defined by $\|\{a_n\}\| = \max |a_n|$. Show that c_0^* is algebraically isomorphic to l^1, where l^1 is the linear space of all absolutely summable sequences $\{b_0, b_1, \dots\}$. HINT: If $f \in c_0^*$, define b_n to be $f(e^n)$, where e^n is the element of c_0 that is 1 in the nth position and 0 elsewhere.

(d) In each of parts a through c, show that the weak and strong topologies are different. Exhibit, in fact, nets (sequences) which converge weakly but not strongly.

(e) Let $X = L^\infty(\mathbb{R})$. For each function $g \in L^1(\mathbb{R})$, define ϕ_g on X by $\phi_g(f) = \int fg$. Show that ϕ_g is an element of X^*. Next, for each finite Borel measure μ on $\mathbb{R}$, define ϕ_μ on X by $\phi_\mu(f) = \int f\,d\mu$. Show that ϕ_μ is an element of X^*. Conclude that, in this sense, $L^1(\mathbb{R})$ is a proper subset of $(L^\infty)^*$.

(f) Let Δ be a second countable locally compact Hausdorff space, and let X be the normed linear space $C_0(\Delta)$ equipped with the supremum norm. Identify X^*.

(g) Let $X_1, \dots, X_n$ be locally convex topological vector spaces. If $X = \bigoplus_{i=1}^n X_i$, show that X^* is isomorphic to $\bigoplus_{i=1}^n X_i^*$.

THEOREM 5.1. (Relation between the Weak and Strong Topologies) *Let $(X, \mathcal{T})$ be a locally convex topological vector space.*

(1) *Let A be a convex subset of X. Then A is strongly closed if and only if it is weakly closed.*
(2) *If A is a convex subset of X, then the weak closure of A equals the strong closure of A.*
(3) *If $\{x_\alpha\}$ is a net in X that converges weakly to an element x, then there exists a net $\{y_\beta\}$, for which each y_β is a (finite) convex combination of some of the x_α's, such that $\{y_\beta\}$ converges strongly to x. If $\mathcal{T}$ is metrizable, then the net $\{y_\beta\}$ can be chosen to be a sequence.*

PROOF. If A is a weakly closed subset, then it is strongly closed since $\mathcal{W} \subseteq \mathcal{T}$. Conversely, suppose that A is a strongly closed convex set and let $x \in X$ be an element not in A. Then, by the Separation Theorem, there exists a continuous linear functional ϕ on X, and a real number s, such that $\phi(y) \leq s$ for all $y \in A$ and $\phi(x) > s$. But then the set $\phi^{-1}(s,\infty)$ is a weakly open subset of X that contains x and is disjoint from A, proving that A is weakly closed, as desired.

If A is a convex subset of X, and if B is the weak closure and C is the strong closure, then clearly $A \subseteq C \subseteq B$. On the other hand, C is convex and strongly closed, hence C is weakly closed. Therefore, $B = C$, and part 2 is proved.

Now let $\{x_\alpha\}$ be a weakly convergent net in X, and let A be the convex hull of the x_α's. If $x = \lim_{\mathcal{W}} x_\alpha$, then x belongs to the weak closure of A, whence to the strong closure of A. Let $\{y_\beta\}$ be a net (sequence if $\mathcal{T}$ is metrizable) of elements of A that converges strongly to x. Then each y_β is a finite convex combination of certain of the x_α's, and part 3 is proved.

DEFINITION. Let X be a locally convex topological vector space, and let X^* be its dual space. For each $x \in X$, define a function $\hat{x}$ on X^* by $\hat{x}(f) = f(x)$. By the *weak* topology* on X^*, we mean the weakest topology $\mathcal{W}^*$ on X^* for which each function $\hat{x}$, for $x \in X$, is continuous.

THEOREM 5.2. (Duality Theorem) *Let $(X, \mathcal{T})$ be a locally convex topological vector space, and let X^* be its dual space. Then:*

(1) *Each function $\hat{x}$ is a linear functional on X^*.*
(2) *$(X^*, \mathcal{W}^*)$ is a locally convex topological vector space. (Each $\hat{x}$ is continuous on $(X^*, \mathcal{W}^*)$.)*
(3) *If ϕ is a continuous linear functional on $(X^*, \mathcal{W}^*)$, then there exists an $x \in X$ such that $\phi = \hat{x}$; i.e., the map $x \to \hat{x}$ is a linear transformation of X onto $(X^*, \mathcal{W}^*)^*$.*
(4) *The map $x \to \hat{x}$ is a topological isomorphism between $(X, \mathcal{W})$ and $((X^*, \mathcal{W}^*)^*, \mathcal{W}^*)$.*

PROOF. If $x \in X$, then

$$\begin{aligned}\hat{x}(af + bg) &= (af + bg)(x) \\ &= af(x) + bg(x) \\ &= a\hat{x}(f) + b\hat{x}(g),\end{aligned}$$

proving part 1.

By the definition of the topology $\mathcal{W}^*$, we see that each $\hat{x}$ is continuous. Also, the set of all functions $\{\hat{x}\}$ separate the points of X^*, for if $f, g \in X^*$, with $f \neq g$, then $f - g$ is not the 0 functional. Hence there exists an $x \in X$ for which $(f - g)(x)$, which is $\hat{x}(f) - \hat{x}(g)$, is not 0. Therefore, the weak topology on X^*, generated by the $\hat{x}$'s, is a locally convex topology. See part c of Exercise 3.11.

Now suppose ϕ is a continuous linear functional on $(X^*, \mathcal{W}^*)$, and let M be the kernel of ϕ. If $M = X^*$, then ϕ is the 0 functional, which is $\hat{0}$. Assume then that there exists an $f \in X^*$, for which $\phi(f) = 1$, whence $f \notin M$. Since ϕ is continuous, M is a closed subset in X^*, and there exists a weak* neighborhood V of f which is disjoint from M. Therefore, by the definition of the topology $\mathcal{W}^*$, there exists a finite set $x_1, \dots, x_n$ of elements of X and a finite set $\epsilon_1, \dots, \epsilon_n$ of positive real numbers such that

$$V = \{g \in X^* : |\hat{x_i}(g) - \hat{x_i}(f)| < \epsilon_i,\ 1 \leq i \leq n\}.$$

Define a map $R : X^* \to \mathbb{R}^n$ by

$$R(g) = (\hat{x_1}(g), \dots, \hat{x_n}(g)).$$

Clearly R is a continuous linear transformation of X^* into $\mathbb{R}^n$. Now $R(f) \notin R(M)$, for otherwise there would exist a $g \in M$ such that $\hat{x_i}(g) = \hat{x_i}(f)$ for all i. But this would imply that $g \in V \cap M$, contradicting the choice of the neighborhood V. Also, $R(M)$ is a subspace of $\mathbb{R}^n$, so contains 0, implying then that $R(f) \neq 0$. Suppose $R(M)$ is of dimension $j < n$. Let $\alpha_1, \dots, \alpha_n$ be a basis for $\mathbb{R}^n$, such that $\alpha_1 = R(f)$ and $\alpha_i \in R(M)$ for $2 \leq i \leq j+1$. We define a linear functional p on $\mathbb{R}^n$ by setting $p(\alpha_1) = 1$ and $p(\alpha_i) = 0$ for $2 \leq i \leq n$.

Now, $p \circ R$ is a continuous linear functional on X^*. If $g \in M$, then $(p \circ R)(g) = p(R(g)) = 0$, since $R(g) \in R(M)$, which is in the span of $\alpha_2, \dots, \alpha_n$. Also, $(p \circ R)(f) = p(R(f)) = 1$, since $R(f) = \alpha_1$. So, $p \circ R$ is a linear functional on X^* which has the same kernel M as ϕ and agrees with ϕ on f. Therefore, $\phi - p \circ R = 0$ everywhere, and $\phi = p \circ R$.

Let $e_1, \dots, e_n$ denote the standard basis for $\mathbb{R}^n$, and let A be the $n \times n$ matrix relating the bases $e_1, \dots, e_n$ and $\alpha_1, \dots, \alpha_n$. That is, $e_i = \sum_{j=1}^n A_{ij}\alpha_j$. Then, if $\alpha = (a_1, \dots, a_n) = \sum_{i=1}^n a_i e_i$, we have

$$\begin{aligned} p(\alpha) &= \sum_{i=1}^n a_i p(e_i) \\ &= \sum_{i=1}^n a_i \sum_{j=1}^n A_{ij} p(\alpha_j) \\ &= \sum_{i=1}^n a_i A_{i1}. \end{aligned}$$

Therefore,

$$\begin{aligned}\phi(g) &= p \circ R(g) \\ &= p(R(g)) \\ &= p((\widehat{x_1}(g), \dots, \widehat{x_n}(g))) \\ &= \sum_{i=1}^n A_{i1}\widehat{x_i}(g) \\ &= (\sum_{i=1}^n A_{i1}\widehat{x_i})(g) \\ &= \widehat{\sum_{i=1}^n A_{i1}x_i}(g) \\ &= \hat{x}(g),\end{aligned}$$

where $x = \sum_{i=1}^n A_{i1}x_i$, and this proves part 3.

We leave the proof of part 4 to the exercises.

EXERCISE 5.3. Prove part 4 of the preceding theorem. HINT: Show that a net $\{x_\alpha\}$ converges in the weak topology of X to an element x if and only if the net $\{\widehat{x_\alpha}\}$ converges in the weak* topology of $(X^*, \mathcal{W}^*)^*$ to the element $\hat{x}$.

DEFINITION. If T is a continuous linear transformation from a locally convex topological vector space X into a locally convex topological vector space Y, we define the *transpose* T^* of T to be the function from Y^* into X^* given by

$$[T^*(f)](x) = f(T(x)).$$

EXERCISE 5.4. If T is a continuous linear transformation from a locally convex topological vector space X into a locally convex topological vector space Y, show that the transpose T^* is a continuous linear transformation from $(Y^*, \mathcal{W}^*)$ into $(X^*, \mathcal{W}^*)$.

EXERCISE 5.5. (Continuous Linear Functionals on Dense Subspaces, Part 1) Let X be a locally convex topological vector space, and let Y be a dense subspace of X.

(a) Prove that Y is a locally convex topological vector space in the relative topology.

(b) Let f be a continuous linear functional on Y, and let x be an element of X that is not in Y. Let $\{y_\alpha\}$ be a net of elements of Y for which $x = \lim y_\alpha$. Prove that the net $\{f(y_\alpha)\}$ is a Cauchy net in $\mathbb{R}$, and hence converges.

(c) Let f be in Y^*. Show that f has a unique extension to a continuous linear functional f' on X. HINT: Show that f' is well-defined and is bounded on a neighborhood of 0.

(d) Conclude that the map $f \to f'$ of part c is an isomorphism of the vector space Y^* onto the vector space X^*. Compare with Exercise 4.11, part a.

EXERCISE 5.6. (Continuous Linear Functionals on Dense Subspaces, Part 2) Let Y be a dense subspace of a locally convex topological vector space X, and equip Y with the relative topology. If Y is a proper subspace of X, show that the map $f \to f'$ of part c of the preceding exercise is not a topological isomorphism of $(Y^*, \mathcal{W}^*)$ and $(X^*, \mathcal{W}^*)$. HINT: Use Theorem 5.2. Compare with part b of Exercise 4.11.

EXERCISE 5.7. (Weak Topologies and Metrizability) Let X be a locally convex topological vector space, and let X^* be its dual space.

(a) Show that the weak topology on X is the weakest topology for which each f_α is continuous, where the f_α's form a basis for the vector space X^*. Similarly, show that the weak* topology on X^* is the weakest topology for which each $\widehat{x_\alpha}$ is continuous, where the x_α's form a basis for the vector space X.

(b) Show that the weak* topology on X^* is metrizable if and only if, as a vector space, X has a countable basis. Show also that the weak topology on X is metrizable if and only if, as a vector space, X^* has a countable basis. HINT: For the "only if" parts, use the fact that in a metric space each point is the intersection of a countable sequence of neighborhoods, whereas, if X has an uncountable basis, then the intersection of any sequence of neighborhoods of 0 must contain a nontrivial subspace of X.

(c) Let X be the locally convex topological vector space $\prod_{n=1}^\infty \mathbb{R}$. Compute X^*, and verify that it has a countable basis. HINT: Show that X^* can be identified with the space of sequences $\{a_1, a_2, \dots\}$ that are eventually 0. That is, as a vector space, X^* is isomorphic to $c_c = \bigoplus_{n=1}^\infty \mathbb{R}$.

(d) Conclude that the topological vector space $X = \prod_{n=1}^\infty \mathbb{R}$ is a Fréchet space that is not normable.

DEFINITION. Let $\mathcal{S}$ be Schwartz space, i.e., the countably normed vector space of Exercise 3.10. Elements of the dual space $\mathcal{S}^*$ of $\mathcal{S}$ are called *tempered distributions* on $\mathbb{R}$.

EXERCISE 5.8. (Properties of Tempered Distributions)

(a) If h is a measurable function on $\mathbb{R}$, for which there exists a positive integer n such that $h(x)/(1+|x|^n)$ is in L^1, we say that h is a *tempered function*. If h is a tempered function, show that the assignment $f \to \int_{-\infty}^{\infty} h(t)f(t)\,dt$ is a tempered distribution u_h. Show further that h is integrable over any finite interval and that the function k, defined by $k(x) = \int_0^x h(t)\,dt$ if $x \geq 0$, and by $k(x) = -\int_x^0 h(t)\,dt$ if $x \leq 0$, also is a tempered function. Show finally that, if g and h are tempered functions for which $u_g = u_h$, then $g(x) = h(x)$ almost everywhere.

(b) Show that $h(x) = 1/x$ is not a tempered function but that the assignment

$$f \to \lim_{\delta \to 0} \int_{|t| \geq \delta} (1/t)f(t)\,dt$$

is a tempered distribution. (Integrate by parts and use the mean value theorem.) Show further that $h(x) = 1/x^2$ is not a tempered function, and also that the assignment

$$f \to \lim_{\delta \to 0} \int_{|t| \geq \delta} (1/t^2)f(t)\,dt$$

is not a tempered distribution. (In fact, this limit fails to exist in general.) In some sense, then, $1/x$ can be considered to determine a tempered distribution but $1/x^2$ cannot.

(c) If μ is a Borel measure on $\mathbb{R}$, for which there exists a positive integer n such that $\int (1/(1+|x|^n))\,d\mu(x)$ is finite, we say that μ is a *tempered measure*. If μ is a tempered measure, show that the assignment $f \to \int_{-\infty}^{\infty} f(t)\,d\mu(t)$ is a tempered distribution u_μ.

(d) Show that the linear functional δ, defined on $\mathcal{S}$ by $\delta(f) = f(0)$ (the so-called Dirac δ-function), is a tempered distribution, and show that $\delta = u_\mu$ for some tempered measure μ.

(e) Show that the linear functional δ', defined on $\mathcal{S}$ by $\delta'(f) = -f'(0)$, is a tempered distribution, and show that δ' is not the same as any tempered distribution of the form u_h or u_μ. HINT: Show that δ' fails to satisfy the dominated convergence theorem.

(f) Let u be a tempered distribution. Define a linear functional u' on $\mathcal{S}$ by $u'(f) = -u(f')$. Prove that u' is a tempered distribution. We call u' the *distributional derivative* of u. As usual, we write $u^{(n)}$ for the nth distributional derivative of u. We have that

$$u^{(n)}(f) = (-1)^n u(f^{(n)}).$$

Verify that if h is a C^∞ function on $\mathbb{R}$, for which both h and h' are tempered functions, then the distributional derivative $(u_h)'$ of u_h coincides with the tempered distribution $u_{h'}$, showing that distributional differentiation is a generalization of ordinary differentiation. Explain why the minus sign is present in the definition of the distributional derivative.

(g) If h is defined by $h(x) = \ln(|x|)$, show that h is a tempered function, that h' is not a tempered function, but that

$$(u_h)'(f) = \lim_{\delta\to 0}\int_{|t|\geq\delta}(1/t)f(t)\,dt = \lim_{\delta\to 0}\int_{|t|\geq\delta}h'(t)f(t)\,dt.$$

Moreover, compute $(u_h)^{(2)}$ and show that it cannot be interpreted in any way as being integration against a function.

(h) If h is a tempered function, show that there exists a tempered function k whose distributional derivative is h, i.e., $u'_k = u_h$.

(i) Suppose h is a tempered function for which the distributional derivative u'_h of the tempered distribution u_h is 0. Prove that there exists a constant c such that $h(x) = c$ for almost all x. HINT: Verify and use the fact that, if $\int_a^b h(x)f(x)\,dx = 0$ for all functions f that satisfy $\int_a^b f(x)\,dx = 0$, then h agrees with a constant function almost everywhere on $[a, b]$.

The next result can be viewed as a kind of Riesz representation theorem for the continuous linear functionals on $\mathcal{S}$.

THEOREM 5.3. (Representing a Tempered Distribution as the Derivative of a Function) *Let u be a tempered distribution. Then there exists a tempered function h and a nonnegative integer N such that u is the Nth distributional derivative $u_h^{(N)}$ of the tempered distribution u_h. We say then that every tempered distribution is the Nth derivative of a tempered function.*

PROOF. Let $u \in \mathcal{S}^*$ be given. Recall that $\mathcal{S}$ is a countably normed space, where the norms $\{\rho_n\}$ are defined by

$$\rho_n(f) = \sup_x \max_{0\leq i,j\leq n}|x^j f^{(i)}(x)|.$$

We see then that $\rho_n \leq \rho_{n+1}$ for all n. Therefore, according to part e of Exercise 3.8, there exists an integer N and a constant M such that $|u(f)| \leq M\rho_N(f)$ for all $f \in \mathcal{S}$. Now, for each $f \in \mathcal{S}$, and each nonnegative integer n, set

$$p_n(f) = \max_{0\leq i,j\leq n}\int_{-\infty}^{\infty}|t^j f^{(i)}(t)|\,dt.$$

There exists a point x_0 and integers i_0 and j_0 such that

$$\begin{aligned}\rho_N(f) &= |x_0^{j_0} f^{(i_0)}(x_0)| \\ &= |\int_{-\infty}^{x_0} j_0 t^{j_0-1} f^{(i_0)}(t) + t^{j_0} f^{(i_0+1)}(t)\,dt| \\ &\leq \int_{-\infty}^{\infty} j_0 |t^{j_0-1} f^{(i_0)}(t) + t^{j_0} f^{(i_0+1)}(t)|\,dt \\ &\leq (N+1) p_{N+1}(f),\end{aligned}$$

showing that $|u(f)| \leq M(N+1)p_{N+1}(f)$ for all $f \in \mathcal{S}$.

Let Y be the normed linear space $\mathcal{S}$, equipped with the norm p_{N+1}. Let

$$X = \bigoplus_{i,j=0}^{N+1} L^1(\mathbb{R}),$$

and define a map $F : Y \to X$ by

$$[F(f)]_{i,j}(x) = x^j f^{(i)}(x).$$

Then, using the max norm on the direct sum space X, we see that F is a linear isometry of Y into X. Moreover, the tempered distribution u is a continuous linear functional on Y and hence determines a continuous linear functional $\tilde{u}$ on the subspace $F(Y)$ of X. By the Hahn-Banach Theorem, there exists a continuous linear functional ϕ on X whose restriction to $F(Y)$ coincides with $\tilde{u}$.

Now $X^* = \bigoplus_{i,j=0}^{N+1} L^\infty(\mathbb{R})$, whence there exist L^∞ functions $v_{i,j}$ such that

$$\phi(g) = \sum_{i=0}^{N+1} \sum_{j=0}^{N+1} \int g_{i,j}(t) v_{i,j}(t)\,dt$$

for all $g = \{g_{i,j}\} \in X$. Hence, for $f \in \mathcal{S}$, we have

$$\begin{aligned} u(f) &= \tilde{u}(F(f)) \\ &= \phi(F(f)) \\ &= \sum_{i=0}^{N+1}\sum_{j=0}^{N+1}\int [F(f)]_{i,j}(t)v_{i,j}(t)\,dt \\ &= \sum_{i=0}^{N+1}\sum_{j=0}^{N+1}\int t^j f^{(i)}(t)v_{i,j}(t)\,dt \\ &= \sum_{i=0}^{N+1}\int f^{(i)}(t)(\sum_{j=0}^{N+1} t^j v_{i,j}(t))\,dt \\ &= \sum_{i=0}^{N+1}\int f^{(i)}(t)v_i(t)\,dt, \end{aligned}$$

where $v_i(t) = \sum_{j=0}^{N+1} t^j v_{i,j}(t)$. Clearly, each v_i is a tempered function, and we let w_i be a tempered function whose $(N+1-i)$th distributional derivative is v_i. (See part h of Exercise 5.8.) Then,

$$\begin{aligned} u(f) &= \sum_{i=0}^{N+1}\int f^{(i)}(t)w_i^{(N+1-i)}(t)\,dt \\ &= \sum_{i=0}^{N+1}(-1)^{N+1-i}\int f^{(N+1)}(t)w_i(t)\,dt \\ &= \int f^{(N+1)}(t)w(t)\,dt, \end{aligned}$$

where $w(t) = \sum_{i=0}^{N+1}(-1)^{N+1-i}w_i$. Hence $u(f) = \int f^{(N+1)}w$, or

$$u = (-1)^{N+1}u_w^{(N+1)} = u_h(N+1),$$

where $h = (-1)^{N+1}w$, and this completes the proof.

DEFINITION. If f is a continuous linear functional on a normed linear space X, define the *norm* $\|f\|$ of f as in Chapter IV by

$$\|f\| = \sup_{\substack{x\in X\\ \|x\|\leq 1}} |f(x)|.$$

DEFINITION. If X is a normed linear space, we define the *conjugate space* of X to be the dual space X^* of X equipped with the norm defined above.

EXERCISE 5.9. Let X be a normed linear space, and let X^* be its dual space. Denote by $\mathcal{W}^*$ the weak* topology on X^* and by $\mathcal{N}$ the topology on X^* defined by the norm.

(a) Show that the conjugate space X^* of X is a Banach space.

(b) Show that, if X is infinite dimensional, then the weak topology on X is different from the norm topology on X and that the weak* topology on the dual space X^* is different from the norm topology on X^*. HINT: Use part c of Exercise 5.1. Note then that the two dual spaces $(X^*, \mathcal{W}^*)^*$ and $(X^*, \mathcal{N})^* \equiv X^{**}$ may be different.

EXERCISE 5.10. (a) Show that the vector space isomorphisms of parts a through c of Exercise 5.2 are isometric isomorphisms.

(b) Let X be a normed linear space and let X^* denote its conjugate space. Let X^{**} denote the conjugate space of the normed linear space X^*. If $x \in X$, define $\hat{x}$ on X^* by $\hat{x}(f) = f(x)$. Show that $\hat{x} \in X^{**}$.

(c) Again let X be a normed linear space and let X^* denote its conjugate space. Prove that $(X^*, \mathcal{W}^*)^* \subseteq X^{**}$; i.e., show that every continuous linear functional on $(X^*, \mathcal{W}^*)$ is continuous with respect to the norm topology on X^*.

(d) Let the notation be as in part b. Prove that the map $x \to \hat{x}$ is continuous from $(X, \mathcal{W})$ into $(X^{**}, \mathcal{W}^*)$.

THEOREM 5.4. *Let X be a normed linear space.*

(1) *If Y is a dense subspace of X, then the restriction map $g \to \tilde{g}$ of X^* into Y^* is an isometric isomorphism of X^* onto Y^*.*
(2) *The weak* topology $\mathcal{W}^*$ on X^* is weaker than the topology defined by the norm on X^*.*
(3) *The map $x \to \hat{x}$ is an isometric isomorphism of the normed linear space X into the conjugate space X^{**} of the normed linear space X^*.*

PROOF. That the restriction map $g \to \tilde{g}$ is an isometric isomorphism of X^* onto Y^* follows from part c of Exercise 5.5 and the definitions of the norms.

If $x \in X$, then $\hat{x}$ is a linear functional on X^* and $|\hat{x}(f)| = |f(x)| \leq \|x\|\|f\|$, showing that $\hat{x}$ is a continuous linear functional in the norm topology of the Banach space X^*, and that $\|\hat{x}\| \leq \|x\|$. Since the weak* topology is the weakest making each $\hat{x}$ continuous, it follows that $\mathcal{W}^*$ is contained in the norm topology on X^*.

Finally, given an $x \in X$, there exists by the Hahn-Banach Theorem an $f \in X^*$ for which $\|f\| = 1$ and $f(x) = \|x\|$. Therefore, $\hat{x}(f) = \|x\|$, showing that $\|\hat{x}\| \geq \|x\|$, and the proof is complete.

EXERCISE 5.11. (The Normed Linear Space of Finite Complex Measures on a Second Countable Locally Compact Hausdorff Space, Part 1) Let Δ be a second countable locally compact Hausdorff space, and let $M(\Delta)$ be the complex vector space of all finite complex Borel measures on Δ. Recall that a finite complex Borel measure on Δ is a map μ of the σ-algebra $\mathcal{B}$ of Borel subsets of Δ into $\mathbb{C}$ satisfying:

(1) $\mu(\emptyset) = 0$.
(2) If $\{E_n\}$ is a sequence of pairwise disjoint Borel sets, then the series $\sum \mu(E_n)$ is absolutely summable and $\mu(\cup E_n) = \sum \mu(E_n)$.

See Exercise 1.12.

(a) If $\mu \in M(\Delta)$, show that μ can be written uniquely as $\mu = \mu_1 + i\mu_2$, where μ_1 and μ_2 are finite signed Borel measures on Δ. Show further that each μ_i may be written uniquely as $\mu_i = \mu_{i1} - \mu_{i2}$, where each μ_{ij} is a finite positive Borel measure, and where μ_{i1} and μ_{i2} are mutually singular.

(b) Let $M(\Delta)$ be as in part a, and let μ be an element of $M(\Delta)$. Given a Borel set E and an $\epsilon > 0$, show that there exists a compact set K and an open set U for which $K \subseteq E \subseteq U$ such that $|\mu(U - K)| < \epsilon$. HINT: Use the fact that Δ is σ-compact, and consider the collection of sets E for which the desired condition holds. Show that this is a σ-algebra that contains the open sets.

EXERCISE 5.12. (The Normed Linear Space of Finite Complex Measures on a Second Countable Locally Compact Hausdorff Space, Part 2) Let $M(\Delta)$ be as in the previous exercise, and for each $\mu \in M(\Delta)$ define

$$\|\mu\| = \sup \sum_{i=1}^{n} |\mu(E_i)|,$$

where the supremum is taken over all partitions $E_1, \ldots, E_n$ of Δ into a finite union of pairwise disjoint Borel subsets.

(a) Show that $\|\mu\| < \infty$.

(b) Prove that $M(\Delta)$ is a normed linear space with respect to the above definition of $\|\mu\|$. This norm is called the *total variation norm.*

(c) If h is a bounded, complex-valued Borel function on Δ, and if $\mu \in M(\Delta)$, show that

$$|\int h\,d\mu| \leq \|h\|_\infty \|\mu\|.$$

HINT: Do this first for simple functions.

(d) For each $\mu \in M(\Delta)$, define a linear functional ϕ_μ on the complex Banach space $C_0(\Delta)$ by

$$\phi_\mu(f) = \int f\,d\mu.$$

Prove that the map $\mu \to \phi_\mu$ is a norm-decreasing isomorphism of the normed linear space $M(\Delta)$ onto $C_0(\Delta)^*$.

(e) Let μ be an element of $M(\Delta)$. Prove that

$$\|\mu\| = \sup \sum_{i=1}^{n} |\mu(K_i)|,$$

where the supremum is taken over all n-tuples $K_1, \ldots , K_n$ of pairwise disjoint compact subsets of Δ.

(f) Let μ be an element of $M(\Delta)$, and let ϕ_μ be the element of $C_0(\Delta)^*$ defined in part d. Prove that $\|\phi_\mu\| = \|\mu\|$. Conclude that $M(\Delta)$ is a Banach space with respect to the norm $\|\mu\|$ and that it is isometrically isomorphic to $C_0(\Delta)^*$.

EXERCISE 5.13. Let X be the normed linear space c_0. See part c of Exercise 5.2.

(a) Compute the conjugate space c_0^*.

(b) Compute c_0^{**} and $(c_0^*, \mathcal{W}^*)^*$. Conclude that $(X^*, \mathcal{W}^*)^*$ can be properly contained in X^{**}; i.e., there can exist linear functionals on X^* that are continuous with respect to the norm topology but not continuous with respect to the weak* topology.

DEFINITION. A Banach space X is called *reflexive* if the map $x \to \hat{x}$, defined in part b of Exercise 5.10, is an (isometric) isomorphism of X onto X^{**}. In general, X^{**} is called the *second dual* or *second conjugate* of X.

EXERCISE 5.14. (Relation among the Weak, Weak*, and Norm Topologies) Let X be a normed linear space. Let $\mathcal{N}$ denote the topology on X^* determined by the norm, let $\mathcal{W}$ denote the weak topology on the locally convex topological vector space $(X^*, \mathcal{N})$, and let $\mathcal{W}^*$ denote the weak* topology on X^*.

(a) If X is finite dimensional, show that all three topologies are the same.

(b) If X is an infinite dimensional reflexive Banach space, show that $\mathcal{W}^* = \mathcal{W}$, and that $\mathcal{W} \subset \mathcal{N}$.

(c) If X is not reflexive, show that $\mathcal{W}^* \subseteq \mathcal{W} \subset \mathcal{N}$.

(d) Let X be a nonreflexive Banach space. Show that there exists a subspace of X^* which is closed in the norm topology $\mathcal{N}$ (whence also in the weak topology $\mathcal{W}$) but not closed in the weak* topology $\mathcal{W}^*$, and conclude then that $\mathcal{W}^* \subset \mathcal{W}$. HINT: Let ϕ be a norm continuous linear functional that is not weak* continuous, and examine its kernel.

(e) Suppose X is an infinite dimensional Banach space. Prove that neither $(X, \mathcal{W})$ nor $(X^*, \mathcal{W}^*)$ is metrizable. HINT: Use the Baire Category Theorem to show that any Banach space having a countable vector space basis must be finite dimensional.

DEFINITION. Let X and Y be normed linear spaces, and let T be a continuous linear transformation from X into Y. The transpose T^* of T is called the *adjoint* of T when it is regarded as a linear transformation from the normed linear space Y^* into the normed linear space X^*.

THEOREM 5.5. *Let T be a continuous linear transformation from a normed linear space X into a normed linear space Y. Then:*

1. *The adjoint T^* of T is a continuous linear transformation of the Banach space Y^* into the Banach space X^*.*
2. *If the range of T is dense in Y, then T^* is 1-1.*
3. *Suppose X is a reflexive Banach space. If T is 1-1, then the range of T^* is dense in X^*.*

PROOF. That T^* is linear is immediate. Further,

$$\begin{aligned}
\|T^*(f)\| &= \sup_{\substack{x\in X\\ \|x\|\le 1}} |[T^*(f)](x)| \\
&= \sup_{\substack{x\in X\\ \|x\|\le 1}} |f(T(x))| \\
&\le \sup_{\substack{x\in X\\ \|x\|\le 1}} \|f\|\|T(x)\| \\
&\le \|f\|\|T\|,
\end{aligned}$$

showing that T^* is continuous in the norm topologies.

If $T^*(f) = 0$, then $f(T(x)) = 0$ for all $x \in X$. If the range of T is dense in Y, then $f(y) = 0$ for all $y \in Y$; i.e., f is the 0 functional, which implies that T^* is 1-1.

Now, if the range of T^* is not dense in X^*, then there exists a nonzero continuous linear functional ϕ on X^* such that ϕ is 0 on the range of T^*. (Why?) Therefore, $\phi(T^*(f)) = 0$ for all $f \in Y^*$. If X is reflexive, then $\phi = \hat{x}$ for some nonzero element $x \in X$. Therefore, $\hat{x}(T^*(f)) = [T^*(f)](x) = f(T(x)) = 0$ for every $f \in Y^*$. But then $T(x)$ belongs to the kernel of every element f in Y^*, whence $T(x)$ is the zero vector, which implies that T is not 1-1. Q.E.D.

THEOREM 5.6. *Let X be a normed linear space, and let $\overline{B}_1$ denote the closed unit ball in the conjugate space X^* of X; i.e., $\overline{B}_1 = \{f \in X^* : \|f\| \leq 1\}$. Then:*

(1) (Alaoglu) *$\overline{B}_1$ is compact in the weak* topology on X^*.*
(2) *If X is separable, then $\overline{B}_1$ is metrizable in the weak* topology.*

PROOF. By the definition of the weak* topology, we have that $\overline{B}_1$ is homeomorphic to a subset of the product space $\prod_{x \in X} \mathbb{R}$. See part e of Exercise 0.8. Indeed, the homeomorphism F is defined by

$$[F(f)]_x = \hat{x}(f) = f(x).$$

Since $|f(x)| \leq \|x\|$, for $f \in \overline{B}_1$, it follows in fact that

$$F(\overline{B}_1) \subseteq \prod_{x \in X} [-\|x\|, \|x\|],$$

which is a compact topological space K. Hence, to see that $\overline{B}_1$ is compact in the weak* topology, it will suffice to show that $F(\overline{B}_1)$ is a closed subset of K. Thus, if $\{f_\alpha\}$ is a net of elements of $\overline{B}_1$, for which the net $\{F(f_\alpha)\}$ converges in K to an element k, then $k_x = \lim [F(f_\alpha)]_x = \lim f_\alpha(x)$, for every x; i.e., the function f on X, defined by $f(x) = k_x$, is the pointwise limit of a net of linear functionals. Therefore f is itself a linear functional on X. Further, $|f(x)| \leq \|x\|$, implying that f is a continuous linear functional on X with $\|f\| \leq 1$, i.e., $f \in \overline{B}_1$. But then, the element $k \in K$ satisfies $k = F(f)$, showing that $F(\overline{B}_1)$ is closed in K, and this proves part 1.

Now, suppose that $\{x_n\}$ is a countable dense subset of X. Then

$$K^* = \prod_n [-\|x_n\|, \|x_n\|]$$

is a compact metric space, and the map $F^* : \overline{B}_1 \to K^*$, defined by

$$[F^*(f)]_n = f(x_n),$$

is continuous and 1-1, whence is a homeomorphism of $\overline{B}_1$ onto a compact metric space, and this completes the proof.

EXERCISE 5.15. (a) Prove that the closed unit ball in L^p is compact in the weak topology, for $1 < p < \infty$.

(b) Show that neither the closed unit ball in $L^1(\mathbb{R})$ nor the closed unit ball in c_0 is compact in its weak topology (or, in fact, in any locally convex vector space topology). HINT: Compact convex sets must have extreme points.

(c) Show that neither $L^1(\mathbb{R})$ nor c_0 is topologically isomorphic to the conjugate space of any normed linear space. Conclude that not every Banach space has a "predual."

(d) Conclude from part c that neither $L^1(\mathbb{R})$ nor c_0 is reflexive. Prove this assertion directly for $L^1(\mathbb{R})$ using part e of Exercise 5.2.

(e) Show that the closed unit ball in an infinite dimensional normed linear space is never compact in the norm topology.

THEOREM 5.7. (Criterion for a Banach Space to Be Reflexive) *Let X be a normed linear space, and let X^{**} denote its second dual equipped with the weak* topology. Then:*

1. *$\hat{X}$, i.e., the set of all $\hat{x}$ for $x \in X$, is dense in $(X^{**}, \mathcal{W}^*)$.*
2. *$\widehat{B_1}$, i.e., the set of all $\hat{x}$ for $\|x\| < 1$, is weak* dense in the closed unit ball V_1 of X^{**}.*
3. *X is reflexive if and only if $\overline{B}_1$ is compact in the weak topology of X.*

PROOF. Suppose $\overline{\hat{X}}$ is a proper subspace of $\underline{(X^{**}, \mathcal{W}^*)}$, and let ϕ be an element of X^{**} that is not in $\overline{\hat{X}}$. Since $\overline{\hat{X}}$ is a closed convex subspace in the weak* topology on X^{**}, there exists a weak* continuous linear functional η on $(X^{**}, \mathcal{W}^*)$ such that $\eta(\hat{x}) = 0$ for all $x \in X$ and $\eta(\phi) = 1$. By Theorem 5.2, every weak* continuous linear functional on X^{**} is given by an element of X^*. That is, there exists an $f \in X^*$ such that

$$\eta(\psi) = \hat{f}(\psi) = \psi(f)$$

for every $\psi \in X^{**}$. Hence,

$$f(x) = \hat{x}(f) = \eta(\hat{x}) = 0$$

for every $x \in X$, implying that $f = 0$. But,

$$\phi(f) = \hat{f}(\phi) = \eta(\phi) = 1,$$

implying that $f \neq 0$. Therefore, we have arrived at a contradiction, whence $\overline{\hat{X}} = X^{**}$ proving part 1.

We show part 2 in a similar fashion. Thus, suppose that $C = \overline{\widehat{B_1}}$ is a proper weak* closed (convex) subset of the norm closed unit ball V_1 of X^{**}, and let ϕ be an element of V_1 that is not an element of C. Again, since C is closed and convex, there exists by the Separation Theorem (Theorem 3.9) a weak* continuous linear functional η on $(X^{**}, \mathcal{W}^*)$ and a real number s such that $\eta(c) \leq s$ for all $c \in C$ and $\eta(\phi) > s$. Therefore, again by Theorem 5.2, there exists an $f \in X^*$ such that

$$\eta(\psi) = \psi(f)$$

for all $\psi \in X^{**}$. Hence,

$$f(x) = \hat{x}(f) = \eta(\hat{x}) \leq s$$

for all $x \in B_1$, implying that

$$|f(x)| \leq s$$

for all $x \in B_1$, and therefore that $\|f\| \leq s$. But, $\|\phi\| \leq 1$, and $\phi(f) = \eta(\phi) > s$, implying that $\|f\| > s$. Again, we have arrived at the desired contradiction, showing that $\widehat{B_1}$ is dense in V_1.

We have seen already that the map $x \to \hat{x}$ is continuous from $(X, \mathcal{W})$ into $(X^{**}, \mathcal{W}^*)$. See part d of Exercise 5.10. So, if $\overline{B}_1$ is weakly compact, then $\widehat{\overline{B}_1}$ is weak* compact in X^{**}, whence is closed in V_1. But, by part 2, $\widehat{\overline{B}_1}$ is dense in V_1, and so must equal V_1. It then follows immediately by scalar multiplication that $\hat{X} = X^{**}$, and X is reflexive.

Conversely, if X is reflexive, then the map $x \to \hat{x}$ is an isometric isomorphism, implying that $V_1 = \widehat{\overline{B}_1}$. Moreover, by Theorem 5.2, the map $x \to \hat{x}$ is a topological isomorphism of $(X, \mathcal{W})$ and $(X^{**}, \mathcal{W}^*)$. Since V_1 is weak* compact by Theorem 5.6, it then follows that $\overline{B_1}$ is weakly compact, and the proof is complete.

EXERCISE 5.16. Prove that every normed linear space is isometrically isomorphic to a subspace of some normed linear space $C(\Delta)$ of continuous functions on a compact Hausdorff space Δ. HINT: Use the map $x \to \hat{x}$.

We conclude this chapter by showing that Choquet's Theorem (Theorem 3.11) implies the Riesz Representation Theorem (Theorem 1.3) for compact metric spaces. Note, also, that we used the Riesz theorem in the proof of Choquet's theorem, so that these two results are really equivalent.

EXERCISE 5.17. (Choquet's Theorem and the Riesz Representation Theorem) Let Δ be a second countable compact topological space, and let $C(\Delta)$ denote the normed linear space of all continuous real-valued functions on Δ equipped with the supremum norm. Let K be the set of all continuous positive linear functionals ϕ on $C(\Delta)$ satisfying $\phi(1) = 1$.

(a) Show that K is compact in the weak* topology of $(C(\Delta))^*$.

(b) Show that the map $x \to \delta_x$ is a homeomorphism of Δ onto the set of extreme points of K. (δ_x denotes the linear functional that sends f to the number $f(x)$.)

(c) Show that every positive linear functional on $C(\Delta)$ is continuous.

(d) Deduce the Riesz Representation Theorem in this case from Choquet's Theorem; i.e., show that every positive linear functional I on $C(\Delta)$ is given by

$$I(f) = \int_\Delta f(x)\, d\mu(x),$$

where μ is a finite Borel measure on Δ.