Kinematic Wave Modeling in the Streams of the Lake Fryxell Basin in the McMurdo Dry Valleys, Antarctica

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Dry Valley Context

- Largest Ice-Free Region
- Approx 4800 km^2
- Extreme Polar Desert
- Minimal Precipitation

USGS Map of McMurdo Dry Valleys located on the coast of Ross Seas and the McMurdo Sound. [Online Image] Available http://wy.water.usgs.gov/projects/antar ctica/htms/map2.htm
USGS Map of Antarctica providing context for McMurdo Dry Valleys location. [Online Image] Available http://wy.water.usgs.gov/projects/antarctica/htms/map1.htm

Taylor Valley Environment

Mean Annual Air Temp:	-16 to -21 °C
Mean Wind Speed:	2 to 4 m/s
Mean Summer Air Temp:	-5 °C

Continuous Daylight Through Summer

Streams:
- Flow for 4 to 12 Weeks
- Unconsolidated Alluvium
- Carry Sediments and Solutes

Moss: only riparian vegetation

- Permanently ice-covered lakes
- Dependence on streams for recharge
- Streams link glaciers and lakes
- Streams are ecosystem hotspots

Witherow, Rebecca. Lake Chad and the Seuss Glacier. McMurdo Dry Valleys Long-Term Ecological Research. 23 July 2008 <http://mcmlter.org/photoDB/DSCN0143.jpg>.

Objectives

- Compare temp regime to hydrograph
- How is water routed?
- “Pure” kinematic wave model
- Understand shape of the wave
- Add-in losses: evaporation and seepage

Gardner, Chris. Canada Glacier Face and Advance. Picasa, 2004-2005 Season. 6 August 2008 [Online Image] <http://picasaweb.google.com/ChristopherBGardner/Antarctica20042005/photo#5102104820509025474 >.

Area of Study

- Taylor Valley
  - 35 km long
  - 3 major lakes
- Lake Fryxell Basin
  - Easternmost Basin
  - 13 streams discharge to lake
  - 5 streams studied

GIS Online Mapping Server Image of Lake Fryxell Basin. [Online Image] Available http://www.mcmlter.org/mapping.htm

Streams of Interest

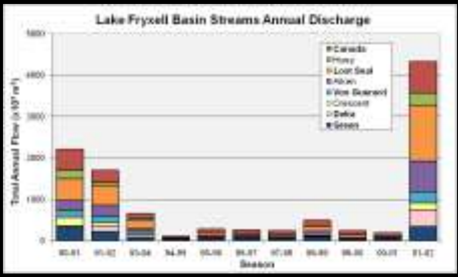
Stream	Length (km)	Width (m)	Gradient (m/m)	Aspect
Canada (F1)	1.5	21.4	0.03	South
Lost Seal (F3)	2.2	4	0.02	South
Von Guerard (F6)	4.9	3-4	0.08	North
Green (F9)	1.2	1-4	0.02	East
Delta (F10)	11.2	10	0.03	North

Variance in Flow

- Daily & Seasonal Variance
- Solar position and glacier face melt
- Incoming shortwave radiation absorption
- Stream Temps:
  - Range: 0 to 25 °C
  - 10°C warmer than air temperature
  - 20 °C variance in one day

Gardner, Chris. Green Creek in Lake Fryxell Basin. McMurdo Dry Valleys Long-Term Ecological Research. 23 July 2008 <http://mcmlter.org/photoDB/green_creek1.jpg>.

Period of Interest



Jaros, Chris. Bar Chart of Total Annual Inflow from Lake Fryxell Basin Streams. From Thesis (2002).

- **One week from each season**
- **Simultaneous flow**

Stream Temperature Regime

Definition: Range and Timing

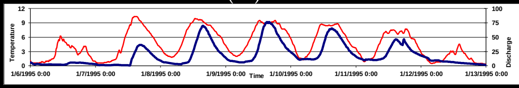
- Driven by solar radiation
- Thermal conductivity is constant
- Compare peaks of temp regime to flow regime

Gardner, Chris. Green Creek in Lake Fryxell Basin. McMurdo Dry Valleys Long-Term Ecological Research. 23 July 2008 <http://mcmlter.org/photoDB/green_creek1.jpg>.

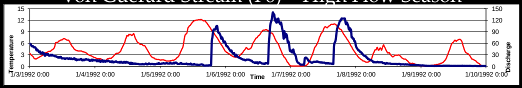
Comparison of Temp and Discharge

-- Temp (°C) -- Discharge (L/s)

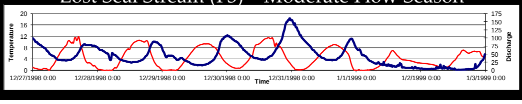
Canada Stream (F1) – Low Flow Season



Von Guerard Stream (F6) – High Flow Season

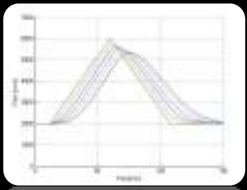


Lost Seal Stream (F3) – Moderate Flow Season



Kinematic Wave Model

- Routes pulse of variable flow
- Function of time & space
- Stream characteristics
- Physical processes
- Boundary Condition:
  $Q(0,t)$
- Initial condition:
  $Q(x,0)$



Kinematic Wave Model

Assumptions Made:

- Neglect acceleration and pressure
- Omit lateral inflow, eddy loss, & wind shear
- Steady and uniform within *dx*
- One-dimensional flow
- Incompressible fluid
- Width less variable than depth
- Uniform width and gradient
- Prismatic channel

St. Venant Equations

Continuity Equation:

$$\frac{d}{dt}\iiint_{c.v.} \rho d\forall + \iint_{c.s.} \rho \mathbf{V} \cdot \mathbf{dA} = 0 \quad \rightarrow \quad \frac{\partial Q}{\partial x} + \frac{\partial A_c}{\partial t} = 0$$

Momentum Equation:

$$\frac{d}{dt}\iiint_{c.v.} \mathbf{V}\rho d\forall + \iint_{c.s.} \mathbf{V}\rho \mathbf{V} \cdot \mathbf{dA} = \sum \mathbf{F}$$

$$\rightarrow \underbrace{\frac{1}{A_c}\frac{\partial Q}{\partial x}}_{\substack{\text{Local}\\\text{Acceleration}}} + \underbrace{\frac{1}{A_c}\frac{\partial}{\partial x}\left(\frac{Q^2}{A_c}\right)}_{\substack{\text{Convective}\\\text{Acceleration}}} + \underbrace{g\frac{\partial y}{\partial x}}_{\substack{\text{Pressure}\\\text{Force}}} - \underbrace{g(S_o}_{\substack{\text{Gravity}\\\text{Force}}} - \underbrace{S_f)}_{\substack{\text{Friction}\\\text{Force}}} = 0$$

For Kinematic Wave

$$\frac{\partial Q}{\partial x} + \frac{\partial A_c}{\partial t} = 0 \quad and \quad S_o = S_f$$

Manning's Equation with $Q = UA_c$ and $R = A_c/P$:

$$Q = \frac{1}{n}\frac{A_c^{5/3}}{P^{2/3}} S_o^{1/2}$$

Solve for $A_C = \alpha Q^{\beta}$ with β = 3/5 and $\alpha = \left(\frac{nB^{2/3}}{\sqrt{S_o}}\right)^{3/5}$

Implicit Finite-Difference Numerical Solution

Differentiate $A_C = \alpha Q^{\beta}$ with respect to time:

$$\frac{\partial A_C}{\partial t} = \alpha\beta Q^{\beta-1}\frac{\partial Q}{\partial t}$$

Substitute into: $\frac{\partial Q}{\partial x} + \frac{\partial A_c}{\partial t} = 0$

$$\frac{\partial Q}{\partial x} + \alpha\beta Q^{\beta-1}\frac{\partial Q}{\partial t} = 0$$

Implicit Finite-Difference Numerical Solution

Value of Q used in $\alpha\beta Q^{\beta-1}$ is found by averaging values:

$$Q \approx \frac{Q_{i+1}^{j} + Q_i^{j+1}}{2}$$

The finite difference form of the linear kinematic wave is:

$$\frac{Q_{i+1}^{j+1} - Q_i^{j+1}}{\Delta x} + \alpha\beta\left(\frac{Q_{i+1}^{j} + Q_i^{j+1}}{2}\right)^{\beta-1}\left(\frac{Q_{i+1}^{j+1} - Q_{i+1}^{j}}{\Delta t}\right) = 0$$

Solved for the unknown , this equation becomes:

$$Q_{i+1}^{j+1} = \frac{\left[\frac{\Delta t}{\Delta x}Q_i^{j+1} + \alpha\beta Q_{i+1}^{j}\left(\frac{Q_{i+1}^{j} + Q_i^{j+1}}{2}\right)^{\beta-1}\right]}{\left[\frac{\Delta t}{\Delta x} + \alpha\beta\left(\frac{Q_{i+1}^{j} + Q_i^{j+1}}{2}\right)^{\beta-1}\right]}$$

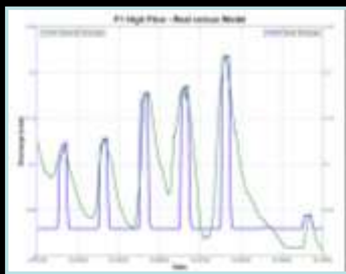
Modeling

- Use of MATLAB
- Lack of observational boundary/initial conditions
- Route water with no losses
- Sine curve provides initial input function
- Timing of direct sun on source
- Position of stream gauge
- Compare model to observed data

Modeling Results

- Captures most wave properties
- Loss of water reflects input function assumption

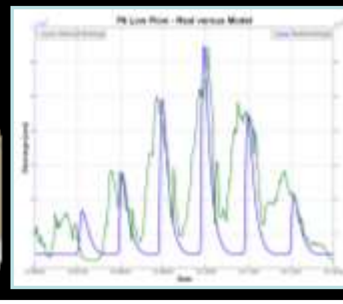
Canada Stream (F1)



Modeling Results

- Evaporation and other losses affect rising limb
- Drainage affects declining limb

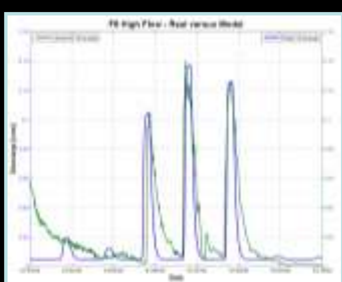
Von Guerard Stream (F6)



Modeling Results

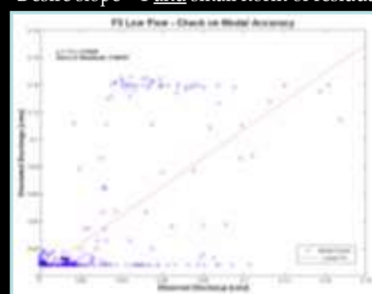
- Works best in longer stream during high flow

Von Guerard Stream (F6)



Model Accuracy

- Plot model output versus observed data
- Desire slope = 1 and small norm of residuals



Von Guerard Stream (F6): Perfect slope, high residuals

Lost Seal Stream (F3): Decent slope, low residuals

Future Work

- Observed data for boundary/initial condition
- Test other input function shapes
- Use glacial melt model input
- Add in evaporative and seepage losses
- Add in drainage gains during low flow
- Route flow backwards from gauge
- See what input function *should* be

Conclusions

Kinematic wave does capture the essential features of diel variation inflow from Lake Fryxell streams

e.g. Kinematic wave correctly models steep rising limb in longer streams at high flow (Von Guerard)

Acknowledgements

- NSF and CU – Boulder Environmental Engineering
- Professor Diane McKnight
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- Josh Koch and Karen Cozzetto
- Audrey Norvell
- Corey Wilson
- Melissa Merrill
- Chris Gardner and Matt Hoffman

Questions?

Derivation of Continuity Eq.

RTT for continuity for unsteady variable-density flow through a control volume:

$$0 = \frac{d}{dt}\iiint_{c.v.} \rho d\forall + \iint_{c.s.} \rho \mathbf{V}\cdot\mathbf{dA}$$

Mass Inflow: $\iint_{Inlet} \rho \mathbf{V}\cdot\mathbf{dA} = -\rho(Q + q dx)$

Mass Outflow: $\iint_{Outlet} \rho \mathbf{V}\cdot\mathbf{dA} = \rho(Q + \frac{\partial Q}{\partial x}dx)$

Rate of Change of Mass Stored in CV: $\frac{d}{dt}\iiint_{c.v.} \rho d\forall = \frac{\partial(\rho A dx)}{\partial t}$

Use substitution, assume constant fluid density and divide by ϱdx:

$$\frac{\partial(\rho A dx)}{\partial t} - \rho(Q + q dx) + \rho(Q + \frac{\partial Q}{\partial x}dx)$$

$$\frac{\partial A}{\partial t} - \frac{Q}{dx} - q + \frac{Q}{dx} + \frac{\partial Q}{\partial x} = 0$$ ← Take no lateral inflow, q

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$ ← Conservative form of St. Venant Continuity Equation

Derivation of Momentum Eq.

RTT for continuity for unsteady nonuniform flow:

$$\sum \mathbf{F} = \frac{d}{dt}\iiint_{c.v.} \mathbf{V}\rho d\forall + \iint_{c.s.} \mathbf{V}\rho\mathbf{V}\cdot\mathbf{dA}$$

Sum of Forces: $\sum \mathbf{F} = F_{gravity} + F_{friction} + F_{eddyloss} + F_{windshear} + F_{pressure}$

$$\sum \mathbf{F} = \rho g A S_o dx - \rho g A S_f dx - \rho g A S_e dx - W_f B \rho g dx - \rho g A \frac{\partial y}{\partial x} dx$$

Momentum Inflow Rate: $\iint_{Inlet} \mathbf{V}\rho\mathbf{V}\cdot\mathbf{dA} = -\rho(\beta V Q + \beta v_x q dx)$

Momentum Outflow Rate: $\iint_{Outlet} \mathbf{V}\rho\mathbf{V}\cdot\mathbf{dA} = \rho\left[\beta V Q + \frac{\partial(\beta V Q)}{\partial x}dx\right]$

$$\beta = \frac{1}{V^2 A}\iint v^2 dA$$

Net Outflow of Momentum Across CS: $\iint_{C.S.} \mathbf{V}\rho\mathbf{V}\cdot\mathbf{dA} = -\rho\left[\beta v_x q + \frac{\partial(\beta V Q)}{\partial x}\right]dx$

Rate of change of momentum stored in the control volume: $\frac{d}{dt}\iiint_{c.v.} \mathbf{V}\rho d\forall = \rho\frac{\partial Q}{\partial t}dx$

Derivation of Momentum Eq.

Substitute into RTT for Momentum:

$$\sum \mathbf{F} = \frac{d}{dt}\iiint_{c.v.} \mathbf{V}\rho d\forall + \iint_{c.s.} \mathbf{V}\rho\mathbf{V}\cdot\mathbf{dA}$$

$$\rho g A S_o dx - \rho g A S_f dx - \rho g A S_e dx - W_f B \rho g dx - \rho g A \frac{\partial y}{\partial x} dx = -\rho\left[\beta v_x q + \frac{\partial(\beta V Q)}{\partial x}\right]dx + \rho\frac{\partial Q}{\partial t}dx$$

Assume constant density & divide through by ρdx, and replace V with Q/A_c:

$$\frac{\partial Q}{\partial t} + \frac{\partial(\beta Q^2/A)}{\partial x} + gA\left(\frac{\partial y}{\partial x} - S_o + S_f + S_e\right) - \beta q v_x + W_f B = 0$$

Neglect lateral inflow, wind shear, and eddy loss. Assume β=1. Use kinematic wave: neglect acceleration and pressure terms:

$$g(S_o - S_f) = 0 \quad \rightarrow \quad S_o = S_f$$

Celerity and Courant Condition

Wave: A variation in flow

Celerity: Velocity with which variation travels downstream

Celerity is different than water velocity

Kinematic wave celerity can be described as:

$$c_k = \frac{dx}{dt} = \frac{dQ}{dA} = \frac{1}{B}\frac{dQ}{dY} \quad \text{where } y \text{ is depth and } dA = B dy$$

Can be solved as: $c_k = \frac{Q^{1-\beta}}{\alpha\beta}$ where $\alpha = \left(\frac{nB^{2/3}}{\sqrt{S_o}}\right)^{3/5}$ and $\beta = 3/5$

Courant Condition: Necessary but insufficient for stability

$$\Delta t \le \frac{\Delta x_i}{c_k}$$

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$$\rightarrow \underbrace{\frac{1}{A_c}\frac{\partial Q}{\partial x}}_{\text{Local Acceleration}} + \underbrace{\frac{1}{A_c}\frac{\partial}{\partial x}\left(\frac{Q^2}{A_c}\right)}_{\text{Convective Acceleration}} + \underbrace{g\frac{\partial y}{\partial x}}_{\text{Pressure Force}} - \underbrace{g\,S_o}_{\text{Gravity Force}} \underbrace{- S_f}_{\text{Friction Force}} = 0$$

For Kinematic Wave

$$\frac{\partial Q}{\partial x}+\frac{\partial A_c}{\partial t}=0 \quad and \quad S_o=S_f$$

Manning's Equation: $U=\frac{C_o}{n}R^{2/3}S_e^{1/2}$

$S_e \approx S_o$, C_o=1, and R = A/P: $U=\frac{1}{n}\left(\frac{A_C}{P}\right)^{2/3}S_o^{1/2}$

With Q = UA: $Q=A_c\frac{1}{n}\left(\frac{A_C}{P}\right)^{2/3}S_o^{1/2} \rightarrow Q=\frac{1}{n}\frac{A_c^{5/3}}{P^{2/3}}S_o^{1/2}$

Solve for $A_C=\alpha Q^{\beta}$ with β = 3/5 and $\alpha=\left(\frac{nP^{2/3}}{\sqrt{S_o}}\right)^{3/5}=\left(\frac{nB^{2/3}}{\sqrt{S_o}}\right)^{3/5}$

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Substitute into: $\frac{\partial Q}{\partial x}+\frac{\partial A_c}{\partial t}=0$

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Solved for the unknown , this equation becomes:

$$Q_{i+1}^{j+1}=\frac{\left[\frac{\Delta t}{\Delta x}Q_i^{j+1}+\alpha\beta Q_{i+1}^{j}\left(\frac{Q_{i+1}^{j}+Q_i^{j+1}}{2}\right)^{\beta-1}\right]}{\left[\frac{\Delta t}{\Delta x}+\alpha\beta\left(\frac{Q_{i+1}^{j}+Q_i^{j+1}}{2}\right)^{\beta-1}\right]}$$

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