

3. DYNAMIC GAMES

We have so far studied simultaneous-move games where players make decisions at the same time. In many economic situations, however, decisions are made at different times by different people. A dynamic game is a game where players move at different points of time. An interesting issue that arises in a dynamic game is whether a player's strategy is sequentially rational, in the sense that it is optimal for the player from any time on. Such considerations may enable us to make more "reasonable" predictions in a game.

3.1 Sequential Rationality and Backward Induction.

Let me start with an example.

Example 3-1. A predation game.

Firm E can choose either In or Out, and Firm I can choose Fight or Accommodate if E has chosen In. The game has two pure-strategy Nash equilibrium: (Out, Fight if E plays In) and (In, Accommodate if E plays In). The trouble with the first NE here is that I 's strategy is not optimal if E were indeed to choose In, in which case it is best for I to choose Accommodate. If E realizes this, then E should choose In, and the first NE would not be a reasonable prediction of the game. This game illustrates a general problem of using NE in dynamic games: a NE may be supported by empty threats that a rational player would not carry out. Nevertheless, such a threat may be a NE strategy since given such a strategy, an opponent may not want to reach the decision node at which the threat is made; but if the decision node is not reached in the play of the game, anything the player says that he will do at this node can be optimal for the player.

To rule out NE that are sustained by strategies that are not optimal starting from some decision node of the game, we use a refinement of NE, called subgame perfect

FIG. 1.

Nash equilibrium (SPNE). The basic idea is that an equilibrium strategy for a player should be optimal for the player starting from any time on in the game (or starting from any decision node of the game tree), given her opponents' strategies. This idea is called the principle of sequential rationality.

For finite games of perfect information, there is a nice procedure to find NE that satisfy sequential rationality. This procedure is called backward induction. Let us again look at Example 3-1. At I 's decision node if E 's has chosen In, it is optimal for I to choose Accommodate. We can then assign payoff $(2, 1)$ to this node and delete the part of the game that follows that node. We then obtain a reduced game where E needs to decide whether to choose Out or In, with payoffs as $(0, 2)$ and $(2, 1)$ respectively. It is clearly optimal then for E to choose In. We have thus found a strategy pair (In, Accommodate if E plays In) that is a NE and also satisfies sequential rationality.

Example 3-2. A three-player finite game of perfect information. Player 1 first chooses between L and R. Player 3 then can choose between l and r if player 1 plays L; and player 2 can choose between a and b if player 1 plays R. Player 3 can choose between l and r if player 1 plays R and player 2 plays either a or b. The payoffs are

FIG. 2.

shown in the game tree.

Using backward induction, we can find a Nash equilibrium that is sequentially rational for each player as: $\sigma_1 = R$, $\sigma_2 = a$ if player 1 plays R , $\sigma_3 = r$ if player 1 plays L , r if player 1 plays R and player 2 plays a , l if player 1 plays R and player 2 plays b .

Thus, for finite games of perfect information, the backward induction procedure first solves optimal actions at the final decision nodes of a game tree. We can then assign to these nodes the payoffs that would be obtained assuming optimal actions after these nodes and delete the part of the game following these decision nodes (Now these are terminal nodes). We then obtain a reduced game. Now repeat the procedure above to the reduced game, and we can obtain another reduced game, and so on, until we have reached the initial decision node.

3.2. Subgame Perfect Nash Equilibrium (SPNE)

For any finite game of perfect information, backward induction will always lead us to a pure strategy Nash equilibrium. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique Nash equilibrium that can be derived

by backward induction. This result is also called the Zermelo's Theorem.

What happens if a game has imperfect information? We may still be able to use backward induction to identify NE that satisfies sequential rationality, but let me first introduce a concept called *subgame*. (As we shall see later, however, for dynamic games of imperfect information, some modifications on the sequential rationality requirement may be needed to obtain sensible predictions.)

Definition 1 *A subgame of an extensive game Γ_E is a subset of the game having the following properties:*

(i) It begins with an information set containing a single decision node, contains all the decisions nodes that are successors (both immediate and late) of this node, and contains only these nodes.

(ii) If decision node x is in the subgame, then every $x' \in H(x)$ is also in the subgame.

Note that the game itself is a subgame.

How many subgames are there in the game of Example 3-2?

Example 3-3. Consider an entry game where firm E can choose either Out or In; and after E 's entry decision is made, if E has chosen In, then both firm E and firm I simultaneously choose Fight or Accommodate.

How many subgames are there in this game?

We can now define a refinement of Nash equilibrium: subgame perfect Nash equilibrium (SPNE).

Definition. A profile of strategies $\sigma = (\sigma_1, \dots, \sigma_I)$ in an I -player extensive form game Γ_E is a SPNE if it induces a Nash equilibrium in every subgame of Γ_E .

Note that a SPNE of a game must be a NE of this game, but a NE may not be a SPNE.

Now if we look back at the game in Example 3-3. The game has three pure strategy

FIG. 3.

Nash equilibria:

(Out, Fight if In), (Fight if E plays In).

(Out, Accommodate if In), (Fight if E plays In).

(In, Accommodate if In), (Accommodate if E plays In).

But only the last one is a SPNE, since in the subgame following In by firm E , the only Nash equilibrium is (Accommodate if In, Accommodate if E plays In). Thus only the strategy profile in the last NE induces a NE in every subgame of the game.

For finite games of perfect information, the strategy profile derived through the backward induction procedure constitutes a Nash equilibrium in the entire game as well as in every subgame. Thus, from Zermeler's theorem, we have:

Proposition 2 *Every finite game of perfect information Γ_E has a pure strategy SPNE. This SPNE can be derived through backward induction. Moreover, this is also the unique pure strategy SPNE if no player has the same payoffs at any two terminal nodes.*

For more general games that may have imperfect information, the set of SPNE can be found through a generalized backward induction procedure as follows:

FIG. 4.

1. Start at the end of the game tree, and identify the Nash equilibria for each of the final subgames.

2. Select one NE in each of the final subgames and derive the reduced game in which these final subgames are replaced by the payoffs that result in these subgames when players use these equilibrium strategies.

3. Repeat steps 1 and 2 for the reduced game. Continue the procedure until the initial decision node is reached. This collection of moves at the various information sets of the game tree constitutes a SPNE.

4. If multiple Nash equilibria are never encountered in any step of this process, then the strategy profile is the unique SPNE. If multiple NE occur in subgames, then the full set of SPNE is found by repeat the process for each NE in each subgame where multiple NE occur.

Example 3-3 has a unique SPNE. The next game has multiple SPNE.

Example 3-4. The Niche choice game. Consider a variation of the entry game where firm E can first choose between Out and In, and if E 's choice is In, both firm E and firm I simultaneously choose SN (small niche) or LN (large niche).

At the subgame starting from E 's decision node following In, there are two pure

strategy Nash equilibria: (SN, LN) and (LN, SN). The SPNE of the full game are:

((Out, SN if In), (LN if E plays In)), ((In, LN if In), (SN if E plays In)).

The next is a game in which a player's strategy set may not be finite.

Example 3-5. This is a simple example of optimal trade policy. Suppose that a Home country firm, H , exports to a foreign market and competes with a foreign firm, F . Both H and E sell only to this foreign market, and they simultaneously choose quantities. The demand in this foreign market is $Q = 100 - p$. Each firm has a constant marginal cost c that is not too large. Before they make the output choices, however, Home country's government can announce and commit to a subsidy for H . That is, for each unit H sells, the government can provide H a subsidy, s , where $-\infty < s < \infty$. Let firm H 's output and profit be q_H and π_H , and firm F 's output and profit be q_F and π_F , Home government's objective is maximize $W = \pi_H - sq_H$. What will be the optimal subsidy s ?

By backward induction, we first find the equilibrium outcomes in the subgame where two firms compete, given each possible choice of s by the government. Since

$$\pi_H = q_H(100 - q_H - q_F) - cq_H + sq_H$$

$$\pi_F = q_F(100 - q_H - q_F) - cq_F$$

The first-order conditions are:

$$100 - 2q_H - q_F - c + s = 0, \quad 100 - 2q_F - q_H - c = 0.$$

We have, in equilibrium,

$$q_H^* = \frac{100 - c + 2s}{3}; \quad q_F^* = \frac{100 - c - s}{3}.$$

$$\begin{aligned} \pi_H^* &= q_H^*(100 - q_H^* - q_F^* - c + s) \\ &= \frac{(100 - c + 2s)^2}{9}. \end{aligned}$$

Therefore, the optimal s maximizes

$$W = \frac{(100 - c + 2s)^2}{9} - s \frac{100 - c + 2s}{3}.$$

We have

$$\frac{dW}{ds} = \frac{4}{9}(100 - c + 2s) - \frac{100 - c + 4s}{3} = 0.$$

or

$$s^* = 25 - \frac{c}{4}.$$

The idea of subgame perfection can also be used when the time horizon is infinite. The next example illustrates this.

Example 3-6. Rubinstein's bargaining game. Two players bargain over how to divide one dollar.. Player 1 first offers $(x, 1 - x)$; player 2 can accept the offer, in which case the game ends with 1 getting x and 2 getting $1 - x$, or player 2 can reject the offer, in which case the game proceeds to period 2 and player 2 can then offer $(y, 1 - y)$, which player 1 can then accept or reject, and if it is rejected, the game proceeds to period 3 and player 1 again makes an offer, and so on. Players' discount factors are δ_1 and δ_2 (i.e., one dollar is worth δ_i ($0 < \delta_i < 1$) after one period for player i).

We can find a SPNE in this game as follows: Suppose at any decision node where it is i 's turn to make an offer, i 's continuation value is v_i . Then a SPNE strategy profile is for player i to always demand getting $v_i = 1 - \delta_j v_j$ whenever she makes an offer, and to accept an offer if and only if she is offered to receive $\delta_i v_i$ or higher, for $i, j = 1, 2$, and $i \neq j$. It remains for us to determine v_i . From the strategy profile, we have

$$v_1 = 1 - \delta_2(1 - \delta_1 v_1).$$

Thus

$$v_1 = \frac{1 - \delta_2}{1 - \delta_1 \delta_2}.$$

Similarly,

$$v_2 = \frac{1 - \delta_1}{1 - \delta_1 \delta_2}.$$

Thus in the SPNE player 1 first offers taking $\frac{1-\delta_2}{1-\delta_1\delta_2}$, leaving $\delta_2 \frac{1-\delta_1}{1-\delta_1\delta_2}$ to player 2. The offer is immediately accepted by player 2.

(Optional reading: Note that if the continuation values of v_1 and v_2 are unique, the prescribed strategies also constitute the unique SPNE. Therefore, to show that the strategy profile is the unique SPNE of the game, we need to show that the continuation values in any SPNE of the game are the same. That is, let \bar{v}_i and \underline{v}_i be the maximum and minimum continuation values of i when it is his turn to make an offer, we need to show $\bar{v}_i = \underline{v}_i$ for $i = 1, 2$.

First, we must have

$$\bar{v}_i \leq 1 - \delta_j \underline{v}_j,$$

since if i offers taking more than $1 - \delta_j \underline{v}_j$, the offer will be rejected, in which case the most i can expect is $\delta_i^2 \bar{v}_i \leq \delta_i (1 - \underline{v}_j)$. Now if $\delta_i^2 \bar{v}_i > 1 - \delta_j \underline{v}_j$, then $1 - \delta_j \underline{v}_j \leq \delta_i (1 - \underline{v}_j)$, or $1 - \delta_i \leq (\delta_j - \delta_i) \underline{v}_j \leq \delta_j - \delta_i$, which is impossible. Thus $\delta_i^2 \bar{v}_i \leq 1 - \delta_j \underline{v}_j$, and so $\bar{v}_i \leq 1 - \delta_j \underline{v}_j$.

Second, since if i offers taking $1 - \delta_j \bar{v}_j$ the offer will be accepted, we have

$$\underline{v}_i \geq 1 - \delta_j \bar{v}_j.$$

We therefore have

$$\bar{v}_i \leq 1 - \delta_j \underline{v}_j \leq 1 - \delta_j (1 - \delta_j \bar{v}_j),$$

or

$$\bar{v}_i \leq \frac{1 - \delta_2}{1 - \delta_1 \delta_2};$$

and

$$\underline{v}_i \geq 1 - \delta_j \bar{v}_j \geq 1 - \delta_j (1 - \delta_i \underline{v}_i),$$

or

$$\underline{v}_i \geq \frac{1 - \delta_2}{1 - \delta_1 \delta_2}.$$

Therefore, $\bar{v}_i = \underline{v}_i$ for $i = 1, 2$.)

An interesting class of games with imperfect information is the repeated play of a simultaneous-move game. Suppose that an extensive form game Γ_E is formed by the repeated play of a simultaneous-move game for T period. If the simultaneous-move game has a unique Nash equilibrium, then game Γ_E has a unique SPNE and it involves all players playing the Nash equilibrium strategies in the simultaneous-move game for each period. If the simultaneous-move game has multiple Nash equilibria, however, then the full game will have multiple SPNE. (You will see this when you do Problem 9.B.9.)

Example 3-5. The repeated Prisoner's dilemma game.

	Prisoner 2	
	DC	C
DC	-2,-2	-10, -1
Prisoner 1		
C	-1, -10	-5, -5

Example 3-6. Rosenthal's Centipede game.

Two players, 1 and 2, each start with one dollar. They alternate saying "Stop" or "Continue". When a player says Continue, one dollar is taken away from her and two dollars are added to her opponents. If a player says Stop, the game ends, and each player's payoff is the amount of dollars she has. If no player says Stop, the game ends when each player's payoff reaches 100 dollars.. Player starts first.

The Centipede game has a unique SPNE: each player says Stop when it is her turn to choose. So the game ends in the first period, and each player receives one dollar each. The prediction of this outcome seems rather unintuitive, and it does not seem to be how people actually behave in such situations. The point of this game

FIG. 5.

is to suggest the limitations of the SPNE concept, and to suggest a different way of thinking such games. If a player may have two types, the type who has the payoffs as in the game above, and the type who actually likes to play continue; and if a player is not sure what type her opponent is, then we might have an equilibrium for both players to choose Continue for some periods. Such considerations started a new types of games: dynamic games of incomplete (imperfect) information.

3.3 Beliefs, Sequential Rationality, and (Weak) Perfect Bayesian Equilibrium

In dynamic games of imperfect information, the notion of subgame perfection may not be much helpful. Consider the following example:

Example 3-7. Suppose an entrant (E) can choose among Out, In_1 , and In_2 . The incumbent (I) chooses between Fight and Accommodate if entry occurs, but I cannot distinguish between In_1 and In_2 .

The game has two pure strategy Nash equilibria: (Out, Fight if entry occurs), and (In_1 , Accommodate if entry occurs). The first one again seems to be supported by empty threats: if entry indeed were to occur, it is always optimal for I to accom-

FIG. 6.

modate. But here subgame perfection would not eliminate this equilibrium. In fact, both equilibria are SPNE since the only subgame in the game is the game itself! In such situations, additional refinement on Nash equilibrium is needed. The basic idea is to require that a player's strategy should be sequentially rational under some belief about what has happened in the game, and the belief should be reasonable in certain sense.

In the literature, there are different suggestions about what reasonable beliefs should be in different applications. It is generally agreed, however, that at the minimum beliefs should be consistent with strategies being played. When only this is required on beliefs, the solution concept is often called weak perfect Bayesian equilibrium (weak PBE). The term "weak" is dropped when some additional restrictions are imposed on beliefs, but sometimes people do not make such distinctions in the literature. That is, sometimes weak PBE is simply called PBE. We now formally define the notions of beliefs and sequential rationality under certain beliefs.

Definition. A system of beliefs μ in extensive form game Γ_E is a specification of a probability $\mu(x) \in [0, 1]$ for each decision node x in Γ_E such that $\sum_{x \in H} \mu(x) = 1$ for all information sets H .

In Example 3-7, under one system of beliefs I assigns equal probabilities to being at the node following In_1 and being at the node following In_2 if entry has occurred. Another belief could be to assign probability 1 and 0 to these two nodes respectively by I should she is called to move. When an information set contains only one node, the player obviously should assigns probability one to being at this node if the information set is reached, and we usually omit this in describing the beliefs.

Now let $E[u_i | H, \mu, \sigma_i, \sigma_{-i}]$ denote player i 's expected payoff starting at her information set H if her beliefs regarding the conditional probabilities of being at the various nodes in H are given by μ , if she follows strategy σ_i , and if her opponents use strategy σ_{-i} .

Definition. A strategy profile $\sigma = (\sigma_1, \dots, \sigma_I)$ in Γ_E is sequentially rational at information set H given a system of beliefs μ if, denoting the player who moves at H by i , we have

$$E[u_i | H, \mu, \sigma_i, \sigma_{-i}] \geq E[u_i | H, \mu, \sigma'_i, \sigma_{-i}]$$

for all $\sigma'_i \in \Delta(S_i)$. If σ satisfies this condition for all information sets H , then we say that σ is sequentially rational given belief μ .

Thus, a strategy profile is sequentially rational if each player is choosing a strategy that is optimal for her at each of her information sets, given her belief and the strategies of other players.

In Example 3-7, suppose that, when I is called to move, I believes that Firm E has chosen In_1 and In_2 each with 50% chance, then strategy profile (In_1 , Accommodate if entry has occurred) is sequentially rational.

We are now ready to define weak perfect Bayesian equilibrium (WPBE).

Definition. A profile of strategies and system of beliefs (σ, μ) is a weak perfect Bayesian equilibrium (weak PBE or WPBE) in extensive form game Γ_E if

- (i) σ is sequentially rational given belief μ ; and
- (ii) μ is derived from σ through Bayes' rule whenever possible.

Condition (ii) says that for any information set H such that $\text{Prob}(H | \sigma) > 0$, then

$$\mu(x) = \frac{\text{Prob}(x | \sigma)}{\text{Prob}(H | \sigma)} \text{ for all } x \in H.$$

On the other hand, there is no restriction on beliefs in information sets that are not reached under the strategy profile. Additional restrictions on these beliefs would lead to notions of PBE and other equilibrium refinements.

In Example 3-7, there is a unique weak PBE, where the strategy profile is (In₁, Accommodate if entry occurs) with I 's belief being that E has chosen In₁ with prob. 1 if entry occurs. To see that there is no other WPBE, notice that under any belief by I sequential rationality must require I to choose Accommodate if entry occurs. Therefore at any WPBE E must choose In₁ with probability 1. But then I must assign prob. 1 to In₁ if entry occurs at any WPBE.

Notice the difference between Nash equilibrium and weak PBE:

A strategy profile σ is a NE iff there exists a system of beliefs μ such that (i) σ is sequentially rational given μ at all information sets H such that $\text{Prob}(H) > 0$; and (ii) μ is derived from σ through Bayes' rule whenever possible.

While for weak PBE, σ needs to be sequentially rational at all information sets given μ . In other words, NE only requires that strategies be sequentially rational along the equilibrium path, while weak PBE requires that strategies be sequentially rational even at information sets that are not reached under the strategy profile.

The game in Example 3-7 is special in that player I has an optimal strategy independent of her beliefs. In general, this is not the case and the trick to find a weak PBE is to find a fixed point such that strategies are sequentially rational given the beliefs and beliefs are consistent with the strategies according to Bayes' rule.

Example 3-8. In this game, it is optimal for I to choose Fight if she believes that In₁ has occurred and to choose Accommodate if she believes In₂ has occurred. Assume $\gamma > 0$.

FIG. 7.

It is easy to show that the game has no pure strategy weak PBE. But the game does have a mixed strategy weak PBE. To see this, suppose E chooses Out, In₁, and In₂ with probabilities σ_0 , σ_1 , and σ_2 respectively; I chooses Fight with prob. σ_F if entry occurs; I 's belief that In₁ has been played when entry occurs is μ_1 .

First, I is willing to choose Fight if and only if

$$-\mu_1 - (1 - \mu_1) \geq -2\mu_1 + (1 - \mu_1),$$

or $\mu_1 \geq \frac{2}{3}$.

Next, if $\mu_1 > \frac{2}{3}$ in any weak PBE, then I must choose Fight with prob. 1. But then E must play In₂ with prob. 1, and the weak PBE would then require $\mu_1 = 0$, a contradiction.

Next, if $\mu_1 < \frac{2}{3}$ in any weak PBE, then I must choose Accommodate with prob. 1. But then E must play In₁ with prob. 1, and the weak PBE would then require $\mu_1 = 1$, again a contradiction.

Thus at any weak PBE $\mu_1 = \frac{2}{3}$. E 's strategy at the weak PBE must then randomize between In₁ and In₂ when choosing to enter the market, with $\sigma_1 = 2\sigma_2$. But then I must randomize between Fight and Accommodate if entry occurs so that E would be

indifferent between In_1 and In_2 . Thus

$$-\sigma_F + 3(1 - \sigma_F) = \gamma\sigma_F + 2(1 - \sigma_F),$$

or $\sigma_F = \frac{1}{2+\gamma}$. This implies that E 's payoff from playing either In_1 or In_2 , given I 's strategy, is $3 - \frac{4}{2+\gamma} > 1$, and thus $\sigma_0 = 0$. Therefore the unique weak PBE is $(\sigma_0, \sigma_1, \sigma_2) = (0, \frac{2}{3}, \frac{1}{3})$, $\sigma_F = \frac{1}{2+\gamma}$, and $\mu_1 = \frac{2}{3}$.

Let us now look at an example where a player may have infinitely many strategies.

Example 3-9. A bargaining model. A seller owns an object that has zero value to him but has value v to a buyer. v can take two possible values: v_H and v_L , with $v_H > v_L > 0$. The value of v is known only to the buyer. The probability of $v = v_H$ is λ , which is common knowledge. In period 1, the seller offers a price to the buyer, and the buyer can either accept or reject the seller's offer. If the buyer accepts, the object is transferred at the price and the game ends. If the buyer rejects the seller's offer, then at period 2 the seller can make another offer and the game ends either the offer is accepted or rejected. The discount factor for both the seller and the buyer is δ . Assume that the buyer always accept the seller's offer if the buyer is indifferent. What are the pure strategy WPBE in this game?

Let S 's strategy be (p_1, p_2) , its belief of B having v_H be μ_1 and μ_2 in the beginning of the first and second periods. First notice that at any WPBE $v_H \geq p_2 \geq v_L$. Given (p_1, p_2) and $v_H \geq p_2 \geq v_L$, the best response for the v_L type is $\sigma_L =$ accept p_1 iff $p_1 \leq v_L$ and accept p_2 iff $p_1 > v_L$ but $p_2 \leq v_L$. The best response for v_H is $\sigma_H =$ accept p_1 iff $p_1 \leq \min\{v_H, v_H - \delta(v_H - p_2)\}$ and accept p_2 iff $p_1 > v_H - \delta(v_H - p_2)$ and $p_2 \leq v_H$. Thus, there can be no WPBE at which both v_H and v_L buy at the second period, because that would imply $p_2 = v_L$ and the seller could be better off offering $p_1 = v_L$. There can also be no WPBE at which only v_H buys at period 2, because that would imply $p_1 > v_H - \delta(v_H - p_2) > v_L$ and $p_2 = v_H$, but the seller would then

be better off offering $p_1 = v_H$. Thus the WPBE is

$$p_1 = v_L \text{ and } p_2 = v_L \text{ or } v_H; (\sigma_L, \sigma_H); \text{ with } \mu_1 = \lambda \text{ and } \mu_2 = 0 \text{ or } 1$$

if

$$v_l \geq \lambda[v_H - \delta(v_H - v_l)] + (1 - \lambda)\delta v_l,$$

or

$$v_l \geq \lambda v_H.$$

The WPBE is

$$p_1 = v_H - \delta(v_H - v_l) \text{ and } p_2 = v_L; (\sigma_L, \sigma_H); \text{ with } \mu_1 = \lambda \text{ and } \mu_2 = 0$$

if

$$v_l \leq \lambda v_H.$$

The first (second) WPBE above, when it exists, is often called a pooling (separating) equilibrium in the literature. We can also simply say what the WPBE strategies are, when we can specify the equilibrium beliefs.

The concept of weak PBE does not impose enough requirement on beliefs in some situations. It is possible that a weak PBE may even not be a subgame perfect Nash equilibrium. Thus the concept of weak PBE may be too weak. Stronger concepts such as perfect Bayesian equilibrium (PBE) and sequential equilibrium have been proposed in the literature. I will spare you from the details of these concepts at this point. (We will revisit some of these concepts later on in Chapter 13. If you come across these concepts in your future research and want to learn more about them, I can give you some references.)