

4. DYNAMIC GAMES

We have so far studied simultaneous-move games where players make decisions at the same time. In many economic situations, however, decisions are made at different times by different people. A dynamic game is a game where players move at different points of time. An interesting issue that arises in a dynamic game is whether a player's strategy is sequentially rational, in the sense that it is optimal for the player from any time on. Such considerations may enable us to make more "reasonable" predictions in a game.

4.1 Sequential Rationality and Backward Induction.

Let me start with an example.

Example 3-1. A predation game.

Firm E can choose either In or Out, and Firm I can choose Fight or Accommodate if E has chosen In. The game has two pure-strategy Nash equilibrium: (Out, Fight if E plays In) and (In, Accommodate if E plays In). The trouble with the first NE here is that I 's strategy is not optimal if E were indeed to choose In, in which case it is best for I to choose Accommodate. If E realizes this, then E should choose In, and the first NE would not be a reasonable prediction of the game. This game illustrates a general problem of using NE in dynamic games: a NE may be supported by empty threats that a rational player would not carry out. Nevertheless, such a threat may be a NE strategy since given such a strategy, an opponent may not want to reach the decision node at which the threat is made; but if the decision node is not reached in the play of the game, anything the player says that he will do at this node can be optimal for the player.

To rule out NE that are sustained by strategies that are not optimal starting from some decision node of the game, we use a refinement of NE, called subgame perfect

FIG. 1.

Nash equilibrium (SPNE). The basic idea is that an equilibrium strategy for a player should be optimal for the player starting from any time on in the game (or starting from any decision node of the game tree), given her opponents' strategies. This idea is called the principle of sequential rationality.

For finite games of perfect information, there is a nice procedure to find NE that satisfy sequential rationality. This procedure is called backward induction. Let us again look at Example 3-1. At I 's decision node if E 's has chosen In, it is optimal for I to choose Accommodate. We can then assign payoff $(2, 1)$ to this node and delete the part of the game that follows that node. We then obtain a reduced game where E needs to decide whether to choose Out or In, with payoffs as $(0, 2)$ and $(2, 1)$ respectively. It is clearly optimal then for E to choose In. We have thus found a strategy pair (In, Accommodate if E plays In) that is a NE and also satisfies sequential rationality.

Example 3-2. A three-player finite game of perfect information. Player 1 first chooses between L and R. Player 3 then can choose between l and r if player 1 plays L; and player 2 can choose between a and b if player 1 plays R. Player 3 can choose between l and r if player 1 plays R and player 2 plays either a or b. The payoffs are

FIG. 2.

shown in the game tree.

Using backward induction, we can find a Nash equilibrium that is sequentially rational for each player as: $\sigma_1 = R$, $\sigma_2 = a$ if player 1 plays R , $\sigma_3 = r$ if player 1 plays L , r if player 1 plays R and player 2 plays a , l if player 1 plays R and player 2 plays b .

Thus, for finite games of perfect information, the backward induction procedure first solves optimal actions at the final decision nodes of a game tree. We can then assign to these nodes the payoffs that would be obtained assuming optimal actions after these nodes and delete the part of the game following these decision nodes (Now these are terminal nodes). We then obtain a reduced game. Now repeat the procedure above to the reduced game, and we can obtain another reduced game, and so on, until we have reached the initial decision node.

4.2. Subgame Perfect Nash Equilibrium (SPNE)

For any finite game of perfect information, backward induction will always lead us to a pure strategy Nash equilibrium. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique Nash equilibrium that can be derived

by backward induction. This result is also called the Zermelo's Theorem.

What happens if a game has imperfect information? We may still be able to use backward induction to identify NE that satisfies sequential rationality, but let me first introduce a concept called *subgame*. (As we shall see later, however, for dynamic games of imperfect information, some modifications on the sequential rationality requirement may be needed to obtain sensible predictions.)

Definition 1 *A subgame of an extensive form game Γ_E is a subset of the game having the following properties:*

(i) It begins with an information set containing a single decision node, contains all the decisions nodes that are successors (both immediate and late) of this node, and contains only these nodes.

(ii) If decision node x is in the subgame, then every $x' \in H(x)$ is also in the subgame.

Note that the game itself is a subgame.

How many subgames are there in the game of Example 3-2?

Example 3-3. Consider an entry game where firm E can choose either Out or In; and after E 's entry decision is made, if E has chosen In, then both firm E and firm I simultaneously choose Fight or Accommodate.

How many subgames are there in this game?

We can now define a refinement of Nash equilibrium: subgame perfect Nash equilibrium (SPNE).

Definition. A profile of strategies $\sigma = (\sigma_1, \dots, \sigma_I)$ in an I -player extensive form game Γ_E is a SPNE if it induces a Nash equilibrium in every subgame of Γ_E .

Note that a SPNE of a game must be a NE of this game, but a NE may not be a SPNE.

Now if we look back at the game in Example 3-3. The game has three pure strategy

FIG. 3.

Nash equilibria:

(Out, Fight if In), (Fight if E plays In).

(Out, Accommodate if In), (Fight if E plays In).

(In, Accommodate if In), (Accommodate if E plays In).

But only the last one is a SPNE, since in the subgame following In by firm E , the only Nash equilibrium is (Accommodate if In, Accommodate if E plays In). Thus only the strategy profile in the last NE induces a NE in every subgame of the game.

For finite games of perfect information, the strategy profile derived through the backward induction procedure constitutes a Nash equilibrium in the entire game as well as in every subgame. Thus, from Zermeler's theorem, we have:

Proposition 2 *Every finite game of perfect information Γ_E has a pure strategy SPNE. This SPNE can be derived through backward induction. Moreover, this is also the unique pure strategy SPNE if no player has the same payoffs at any two terminal nodes.*

For more general games that may have imperfect information, the set of SPNE can be found through a generalized backward induction procedure as follows:

FIG. 4.

1. Start at the end of the game tree, and identify the Nash equilibria for each of the final subgames.

2. Select one NE in each of the final subgames and derive the reduced game in which these final subgames are replaced by the payoffs that result in these subgames when players use these equilibrium strategies.

3. Repeat steps 1 and 2 for the reduced game. Continue the procedure until the initial decision node is reached. This collection of moves at the various information sets of the game tree constitutes a SPNE.

4. If multiple Nash equilibria are never encountered in any step of this process, then the strategy profile is the unique SPNE. If multiple NE occur in subgames, then the full set of SPNE is found by repeat the process for each NE in each subgame where multiple NE occur.

Example 3-3 has a unique SPNE. The next game has multiple SPNE.

Example 3-4. The Niche choice game. Consider a variation of the entry game where firm E can first choose between Out and In, and if E 's choice is In, both firm E and firm I simultaneously choose SN (small niche) or LN (large niche).

At the subgame starting from E 's decision node following In, there are two pure

strategy Nash equilibria: (SN, LN) and (LN, SN). The SPNE of the full game are:

((Out, SN if In), (LN if E plays In)), ((In, LN if In), (SN if E plays In)).

The next is a game in which a player's strategy set may not be finite.

Example 3-5. This is a simple example of optimal trade policy. Suppose that a Home country firm, H , exports to a foreign market and competes with a foreign firm, F . Both H and E sell only to this foreign market, and they simultaneously choose quantities. The demand in this foreign market is $Q = 100 - p$. Each firm has a constant marginal cost c that is not too large. Before they make the output choices, however, Home country's government can announce and commit to a subsidy for H . That is, for each unit H sells, the government can provide H a subsidy, s , where $-\infty < s < \infty$. Let firm H 's output and profit be q_H and π_H , and firm F 's output and profit be q_F and π_F , Home government's objective is maximize $W = \pi_H - sq_H$. What will be the optimal subsidy s ?

By backward induction, we first find the equilibrium outcomes in the subgame where two firms compete, given each possible choice of s by the government. Since

$$\begin{aligned}\pi_H &= q_H(100 - q_H - q_F) - cq_H + sq_H \\ \pi_F &= q_F(100 - q_H - q_F) - cq_F\end{aligned}$$

The first-order conditions are:

$$100 - 2q_H - q_F - c + s = 0, \quad 100 - 2q_F - q_H - c = 0.$$

We have, in equilibrium,

$$\begin{aligned}q_H^* &= \frac{100 - c + 2s}{3}; \quad q_F^* = \frac{100 - c - s}{3} \\ \pi_H^* &= q_H^*(100 - q_H^* - q_F^* - c + s) \\ &= \frac{(100 - c + 2s)^2}{9}.\end{aligned}$$

Therefore, the optimal s maximizes

$$W = \frac{(100 - c + 2s)^2}{9} - s \frac{100 - c + 2s}{3}.$$

We have

$$\frac{dW}{ds} = \frac{4}{9}(100 - c + 2s) - \frac{100 - c + 4s}{3} = 0.$$

or

$$s^* = 25 - \frac{c}{4}.$$

The idea of subgame perfection can also be used when the time horizon is infinite. The next example illustrates this.

Example 3-6. Rubinstein's bargaining game. Two players bargain over how to divide one dollar.. Player 1 first offers $(x, 1 - x)$; player 2 can accept the offer, in which case the game ends with 1 getting x and 2 getting $1 - x$, or player 2 can reject the offer, in which case the game proceeds to period 2 and player 2 can then offer $(y, 1 - y)$, which player 1 can then accept or reject, and if it is rejected, the game proceeds to period 3 and player 1 again makes an offer, and so on. Players' discount factors are δ_1 and δ_2 (i.e., one dollar is worth δ_i ($0 < \delta_i < 1$) after one period for player i).

We can find a SPNE in this game as follows: Suppose at any decision node where it is i 's turn to make an offer, i 's continuation value is v_i . Then a SPNE strategy profile is for player i to always demand getting $v_i = 1 - \delta_j v_j$ whenever she makes an offer, and to accept an offer if and only if she is offered to receive $\delta_i v_i$ or higher, for $i, j = 1, 2$, and $i \neq j$. It remains for us to determine v_i . From the strategy profile, we have

$$v_1 = 1 - \delta_2(1 - \delta_1 v_1).$$

Thus

$$v_1 = \frac{1 - \delta_2}{1 - \delta_1 \delta_2}.$$

Similarly,

$$v_2 = \frac{1 - \delta_1}{1 - \delta_1 \delta_2}.$$

Thus in the SPNE player 1 first offers taking $\frac{1-\delta_2}{1-\delta_1\delta_2}$, leaving $\delta_2 \frac{1-\delta_1}{1-\delta_1\delta_2}$ to player 2. The offer is immediately accepted by player 2.

(Optional reading: Note that if the continuation values of v_1 and v_2 are unique, the prescribed strategies also constitute the unique SPNE. Therefore, to show that the strategy profile is the unique SPNE of the game, we need to show that the continuation values in any SPNE of the game are the same. That is, let \bar{v}_i and \underline{v}_i be the maximum and minimum continuation values of i when it is his turn to make an offer, we need to show $\bar{v}_i = \underline{v}_i$ for $i = 1, 2$.

First, we must have

$$\bar{v}_i \leq 1 - \delta_j \underline{v}_j,$$

since if i offers taking more than $1 - \delta_j \underline{v}_j$, the offer will be rejected, in which case the most i can expect is $\delta_i^2 \bar{v}_i \leq \delta_i(1 - \underline{v}_j)$. Now if $\delta_i^2 \bar{v}_i > 1 - \delta_j \underline{v}_j$, then $1 - \delta_j \underline{v}_j \leq \delta_i(1 - \underline{v}_j)$, or $1 - \delta_i \leq (\delta_j - \delta_i) \underline{v}_j \leq \delta_j - \delta_i$, which is impossible. Thus $\delta_i^2 \bar{v}_i \leq 1 - \delta_j \underline{v}_j$, and so $\bar{v}_i \leq 1 - \delta_j \underline{v}_j$.

Second, since if i offers taking $1 - \delta_j \bar{v}_j$ the offer will be accepted, we have

$$\underline{v}_i \geq 1 - \delta_j \bar{v}_j.$$

We therefore have

$$\bar{v}_i \leq 1 - \delta_j \underline{v}_j \leq 1 - \delta_j(1 - \delta_j \bar{v}_j),$$

or

$$\bar{v}_i \leq \frac{1 - \delta_2}{1 - \delta_1 \delta_2};$$

and

$$\underline{v}_i \geq 1 - \delta_j \bar{v}_j \geq 1 - \delta_j(1 - \delta_i \underline{v}_i),$$

or

$$\underline{v}_i \geq \frac{1 - \delta_2}{1 - \delta_1 \delta_2}.$$

Therefore, $\bar{v}_i = \underline{v}_i$ for $i = 1, 2$.)

An interesting class of games with imperfect information is the repeated play of a simultaneous-move game. Suppose that an extensive form game Γ_E is formed by the repeated play of a simultaneous-move game for T period. If the simultaneous-move game has a unique Nash equilibrium, then game Γ_E has a unique SPNE and it involves all players playing the Nash equilibrium strategies in the simultaneous-move game for each period. If the simultaneous-move game has multiple Nash equilibria, however, then the full game will have multiple SPNE. (You will see this when you do Problem 9.B.9.)

Example 3-5. The repeated Prisoner's dilemma game.

	Prisoner 2	
	DC	C
DC	-2,-2	-10, -1
Prisoner 1		
C	-1, -10	-5, -5

Example 3-6. Rosenthal's Centipede game.

Two players, 1 and 2, each start with one dollar. They alternate saying "Stop" or "Continue". When a player says Continue, one dollar is taken away from her and two dollars are added to her opponents. If a player says Stop, the game ends, and each player's payoff is the amount of dollars she has. If no player says Stop, the game ends when each player's payoff reaches 100 dollars.. Player starts first.

The Centipede game has a unique SPNE: each player says Stop when it is her turn to choose. So the game ends in the first period, and each player receives one dollar each. The prediction of this outcome seems rather unintuitive, and it does not seem to be how people actually behave in such situations. The point of this game

FIG. 5.

is to suggest the limitations of the SPNE concept, and to suggest a different way of thinking such games. If a player may have two types, the type who has the payoffs as in the game above, and the type who actually likes to play continue; and if a player is not sure what type her opponent is, then we might have an equilibrium for both players to choose Continue for some periods. Such considerations started a new types of games: dynamic games of incomplete (imperfect) information.