1 Competitive Equilibrium with Complete Markets

In this part, we study a pure exchange economy with infinite horizon and Markov endowments. We will first study the individual decision under a certain market specification which will be referred to as the Arrow-Debreu structure named after Kenneth Arrow and Gerard Debreu. This involves constructing a setting in which all securities trade takes place at time 0. Then we shall introduce an alternative specification involving sequential trading of one period Arrow securities. Both specifications are referred to as complete markets even though the underlying timing assumptions and asset structures are quite different.

1.1 Complete Markets with Arrow-Debreu Securities

We shall first specify the structure of the uncertainty under which we will define the competitive equilibrium. Let s_t denote the state at time t. We assume that trading occurs after the initial state s_0 is realized. Let $\pi(s' \mid s)$ be a Markov chain with initial distribution $\pi_0(s)$ given. Also let's denote the history of events until time t as the sequence $s^t = \{s_t, s_{t-1}, ..., s_0\}$. Therefore the unconditional probability of the realization of a particular history s^t is given by

$$\pi(s^t) = \pi(s_t \mid s_{t-1})\pi(s_{t-1} \mid s_{t-2}).....\pi(s_1 \mid s_0)\pi(s_0).$$

Likewise, the probability of s^t conditional on the time 0 state s_0 is given by

$$\pi(s^t \mid s_0) = \pi(s_t \mid s_{t-1})\pi(s_{t-1} \mid s_{t-2}).....\pi(s_1 \mid s_0).$$

Note that the Markov assumption implies the the probability of s^t conditional on having the history s^τ at time τ, $\pi(s^t \mid s^\tau) = \pi(s^t \mid s_\tau)$. In other words, this probability does not depend on the history $s^{\tau-1}$.

Suppose that there are I individuals indexed by $i \in \{1, 2, ..., I\}$. Every individual is endowed with an uncertain number of goods at each period. The number of goods the i^{th} individual receives at time t depends on s_t and is denoted with $y^i(s_t)$ and s_t is common knowledge (publicly observable). Given this individual-specific stochastic endowment process and an infinite planning horizon, every individual purchases a history-dependent consumption plan $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$ to maximize her expected life-time utility given by

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t^i(s^t)]\pi(s^t \mid s_0). \qquad (1)$$

The utility function $u(.)$ is increasing, strictly concave and twice continuously differentiable. We also retain the assumption $\lim_{c \to 0} u'(c) = \infty$. Note that households are trading history-specific consumption goods. In other words, they trade dated state-contingent claims to consumption. All trades occur at time 0 after the initial state is realized. The price of the history-specific consumption at time t, $c_t(s^t)$, is denoted by $q_t^0(s^t)$. The superscript refers to the date at which trades occur and the subscript refers to the date at which deliveries are to be made. Therefore, the budget constraint can be written as

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t^i(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y^i(s_t). \tag{2}$$

The household problem now can be defined as maximization of (1) subject to (2) and $c_t^i(s^t) \geq 0$ for all t and s^t given the prices $\{q_t^0(s^t)\}_{t=0}^{\infty}$. The first order conditions of this problem give

$$U'(c^i) = \mu^i q_t^0(s^t)$$

where μ^i is the Lagrange multiplier of the i^{th} individual. More explicitly,

$$\beta^t u'[c_t^i(s^t)]\pi(s^t \mid s_0) = \mu^i q_t^0(s^t). \tag{3}$$

Definition 1 *A price system is a sequence of functions $\{q_t^0(s^t)\}_{t=0}^{\infty}$. An allocation is a list of sequences of functions $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$ for all $i \in \{1, 2, ..., I\}$. An allocation is feasible if it satisfies*

$$\sum_i y^i(s_t) \geq \sum_i c_t^i(s^t). \tag{4}$$

Definition 2 *A competitive equilibrium is a feasible allocation and price system such that the allocation solves each household's problem.*

Note that equation (3) implies

$$\frac{u'[c_t^i(s^t)]}{u'[c_t^j(s^t)]} = \frac{\mu^i}{\mu^j}. \tag{5}$$

Proposition 1 *The competitive equilibrium allocation is not history dependent, that is $c_t^i(s^t) = c_t^i(s_t)$.*

Proof. It follows from (5) that for any $i, j \in \{1, 2, ..., I\}$,

$$c_t^j(s^t) = g\left(c_t^i(s^t); \frac{\mu^i}{\mu^j}\right).$$

The left hand side of the feasibility constraint (4) does not depend on the entire history of s. It only depends on s_t. Therefore (4) and (5) imply that $c_t^i(s^t) = c_t^i(s_t)$ for all $i \in \{1, 2, ..., I\}$.When the utility function is CRRA (Constant Relative Risk Aversion), that is $u(c_t) = (1-\sigma)^{-1}c_t^{1-\sigma}$ with $\sigma > 0$, the first order conditions of the household problem gives:

$$\frac{c_t^i(s^t)}{c_t^j(s^t)} = \left(\frac{\mu^i}{\mu^j}\right)^{-\frac{1-}{\sigma}}. \tag{6}$$

■

Equation (6) reveals that the consumption ratio of any two agents remain constant regardless of time and history. In other words, the consumption fractions assigned to each individual are independent of the realization of s_t and also stays constant over time. (But keep in mind that this by no means imply that the consumption levels would also remain constant.) Equation (6) tells us that individual consumption is perfectly correlated with aggregate output and consumption.

Example 1 *Let $s_t \in [0,1]$ be a Markov process. Consider two households with $y_t^1 = s_t$ and $y_t^2 = 1 - s_t$. Let's find the equilibrium for this simple economy. First, note that aggregate output is constant, that is $\sum_i y_t^i = 1$. We already know from (6) that the consumption ratio of the individuals must be constant. Since the aggregate output is also constant, it is reasonable to conjecture that consumption levels for both individuals are constant and history-independent. Therefore, let's guess that $c_t^i = c_0^i$ for all i. Then it follows from the first order conditions that*

$$q_0^t(s^t) = \beta^t \pi(s^t \mid s_0) \frac{u'(c_0^i)}{\mu^i}$$

and given the time 0 budget constraint we find

$$\frac{u'(c_0^i)}{\mu^i} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t \mid s_0)[c_0^i - y^i(s^t)] = 0.$$

The above equation can be solved to find the conjectured level of consumption as follows:

$$c_0^i = (1-\beta) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t \mid s_0) y^i(s^t) \tag{7}$$

Summing this equation over i gives $c_0^1 + c_0^2 = 1$.

A couple of observations are in order. First, note that since $\sum_i y^i(s^t) = 1$ there is no aggregate uncertainty. In other words, uncertainty only governs to the individual endowment processes. Second, the existence of a complete set of stage-contingent claims market, by making perfect risk-sharing available, renders possible constant patterns of consumption for both agents. Now let's drop no aggregate uncertainty assumption and observe how our result ceases to obtain. Suppose that $y_t^1 = s_t$ and $y_t^2 = a_t - s_t$ where $\ln a_t \tilde{} N(0,1)$ and s_t is as defined above. In this case, if we conjecture that $c_t^i = c_0^i$ for all i than it follows from (7) that $\sum_i c_0^i = 1$ which is not necessarily smaller than or equal to a_t. Hence, this result violates the feasibility constraint. It follows that optimal consumption patterns cannot remain constant in the presence of aggregate uncertainty.

1.2 Arrow Securities

We have assumed so far that all trades take place in period 0 in a massive state-contingent claims market and all deliveries are made accordingly afterwards.

In this section we shall adopt a sequential approach and present an alternative framework for state-contingent claims trade.

Before we proceed to the sequential trading of state-contingent claims, it will be useful to construct a preliminary asset pricing framework which will later be discussed in detail in the last part of the course. Consider an asset which promises to deliver a stream of claims $\{d(s_t)\}_{t=0}^{\infty}$ at time t, state s^t consumption. Then, the price of this asset must be given by

$$a_0^0 = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) d(s_t) \tag{8}$$

otherwise there would be an arbitrage opportunity. In other words, if the price is different from the above expression, one could make unbounded profits by synthesizing this asset through purchases or sales of state-contingent dated commodities and then either buying or selling the composite asset. This type of assets are referred to as *redundant assets* since they can be priced simply by accounting for the state-contingent dated claims which are already priced in a complete markets environment.

Note that (8) expresses the price of the asset in date-0 and state-s_0 goods. Now reconsider the stream $\{d(s_t)\}_{t=0}^{\infty}$ and suppose that for some $\tau \geq 1$ we strip off the first $\tau - 1$ periods of the stream and want to get the time-0 value of the remaining stream. The price of this constructed asset is given by

$$a_\tau^0(s^\tau) = \sum_{t=\tau}^{\infty} \sum_{\{\widetilde{s}^t : \widetilde{s}^\tau = s^\tau\}} q_t^0(\widetilde{s}^t) d(\widetilde{s}_t).$$

To convert this price into units of time τ and state s^τ consumption goods, divide by $q_\tau^0(s^\tau)$ to obtain

$$a_\tau^\tau(s^\tau) = \frac{a_\tau^0(s^\tau)}{q_\tau^0(s^\tau)} = \sum_{t=\tau}^{\infty} \sum_{\{\widetilde{s}^t : \widetilde{s}^\tau = s^\tau\}} \frac{q_t^0(\widetilde{s}^t)}{q_\tau^0(s^\tau)} d(\widetilde{s}_t).$$

Note that

$$\frac{q_t^0(s^t)}{q_\tau^0(s^\tau)} = q_t^\tau(s^t) = \beta^{t-\tau} \frac{u'(c_t^i(s^t))}{u'(c_\tau^i(s^\tau))} \pi(s^t \mid s^\tau) \tag{9}$$

where $q_t^\tau(s^t)$ is the price of one unit of consumption delivered at time t, state s^t in terms of the date-τ, state s^τ consumption good. Thus, the price of this asset at time τ in state s^τ consumption good is given by

$$a_\tau^\tau(s^\tau) = \sum_{t=\tau}^{\infty} \sum_{\{\widetilde{s}^t : \widetilde{s}^\tau = s^\tau\}} q_t^\tau(\widetilde{s}^t) d(\widetilde{s}_t).$$

Proposition 2 *The equilibrium price of date-t, state-s^t consumption goods expressed in terms of date τ ($0 \leq \tau \leq t$), state s^τ consumption goods is not history-dependent: $q_t^\tau(s^t) = q_k^j(\widetilde{s}^k)$ for $j, k \geq 0$ such that $t - \tau = k - j$ and $[s_t, s_{t-1}, ..., s_\tau] = [\widetilde{s}_k, \widetilde{s}_{k-1}, ..., \widetilde{s}_j]$.*

When $t = \tau + 1$, (9) gives

$$q^{\tau}_{\tau+1}(s^{\tau+1}) = \beta \frac{u'(c^i_{\tau+1}(s^{\tau+1}))}{u'(c^i_{\tau}(s^{\tau}))} \pi(s^{\tau+1} \mid s^{\tau}).$$

The right hand side of the above expression is the one period *pricing kernel.*

Note that a household's wealth at time-t is given by the value of all its current and future net claims. This value, when expressed in terms of the date-t, state s^t consumption good is given by

$$\Upsilon^i_t(s^t) = \sum_{\tau=t}^{\infty} \sum_{\{\widetilde{s}^{\tau}:\widetilde{s}^t = s^t\}} q^t_{\tau}(\widetilde{s}^{\tau})[c^i_{\tau}(\widetilde{s}^{\tau}) - y^i(\widetilde{s}_{\tau})]. \tag{10}$$

Also notice that the feasibility constraint implies that $\sum_i \Upsilon^i_t(s^t) = 0$.

In a sequential trading economy, we assume there is a sequence of markets in one-period-ahead state-contingent claims to wealth or consumption. These markets reopen every period and at each date t households trade claims to date $t+1$ consumption, whose payment is contingent on the realization of $s_{t+1.}$ Let θ^i_t denote the claims to time-t consumption that household brings into time t. Suppose that $Q(s_{t+1} \mid s_t)$ is a one-period pricing kernel. It gives the price of one unit of claim to time-$t+1$ consumption contingent on the state at $t+1$ being s_{t+1}, given that the state at t is s_t. Then the sequential budget constraint of the i^{th} household can be constructed as follows:

$$c^i_t + \sum_{s_{t+1}} \theta^i_{t+1}(s_{t+1}) Q(s_{t+1} \mid s_t) \leq y^i(s_t) + \theta^i_t \tag{11}$$

We can restate the household problem as maximization of (1) subject to (11). However, we also need a restriction on the amount of debt a household can accumulate. Otherwise, households would simply run Ponzi schemes by borrowing and paying back again by borrowing even more. We impose the weakest possible restriction to rule out Ponzi schemes. We require that households never promise to pay more than their endowment sequence simply because it will not be feasible for them to do so. Consider the "tail" of household i's endowment sequence at time τ in state s^{τ} :

$$\begin{aligned} A^i(s^{\tau}) &= \sum_{t=\tau}^{\infty} \sum_{\{\widetilde{s}^t:\widetilde{s}^{\tau} = s^{\tau}\}} q^t_{\tau}(\widetilde{s}^{\tau}) y^i(\widetilde{s}_{\tau}) \\ &= \sum_{\tau=t}^{\infty} \sum_{\{\widetilde{s}^{\tau}:\widetilde{s}^t = s^t\}} \beta^{\tau-t} \frac{u'(c^i_{\tau}(\widetilde{s}^{\tau}))}{u'(c^i_t(\widetilde{s}^t))} \pi(\widetilde{s}^{\tau} \mid \widetilde{s}^t) y^i(\widetilde{s}_{\tau}) \\ &\equiv \overline{A}^i(s_t) \end{aligned} \tag{12}$$

The above expression gives the value of the maximum amount that household i can repay starting from state s^t, assuming that her consumption is zero forever.

Now the household problem can be defined as maximization of (1) subject to (11) and

$$-\theta^i_{t+1}(s_{t+1}) \leq \overline{A}^i(s_{t+1}). \tag{13}$$

The Bellman equation for the household's problem is given by

$$v^i(\theta^i_t, s_t) = \max_{c^i_t, \theta^i_{t+1}(s_{t+1})} \left\{ u(c^i_t) + \beta \sum_{s_{t+1}} v^i(\theta^i_{t+1}(s_{t+1}), s_{t+1}) \pi(s_{t+1} \mid s_t) \right\} \tag{14}$$

subject to (11) , (13) as well as $c^i_t \geq 0$. The optimal decision rules that follow from this problem are of following forms:

$$\begin{aligned} c^i_t &= h^i(\theta^i_t, s_t) \\ \theta^i_{t+1}(s_{t+1}) &= g^i(\theta^i_t, s_t, s_{t+1}) \end{aligned}$$

Definition 3 *A distribution of wealth is a vector* $\Theta_t = \left\{\theta^i_t\right\}_{i=1}^I$ *satisfying* $\sum_i \theta^i_t = 0$.

Definition 4 *A recursive competitive equilibrium is an initial distribution of wealth,* Θ_0*, a pricing kernel* $Q(s_{t+1} \mid s_t)$*, value functions* $\left\{v^i(\theta^i_t, s_t)\right\}_{i=1}^I$ *and decision rules* $\left\{h^i(\theta^i_t, s_t), g^i(\theta^i_t, s_t, s_{t+1})\right\}_{i=1}^I$ *such that for all i, given the initial wealth distribution and pricing kernel the decision rules solve the household problem and for all realizations of* $\{s_t\}_{t=0}^\infty$*, the consumption and asset portfolios implied by the decision rules satisfy* $\sum_i c^i_t = \sum_i y^i(s_t)$ *and* $\sum_i \theta^i_{t+1}(s_{t+1}) = 0$ *for all* t *and* s_t*.*

One can solve (14) subject to (12) to find

$$Q(s_{t+1} \mid s_t) = \beta \frac{u'(c^i_{t+1}(s^{t+1}))}{u'(c^i_t(s^t))} \pi(s^{t+1} \mid s^t) \tag{15}$$

where $c^t_i = h^i(\theta^i_t, s_t)$, $\theta^i_{t+1}(s_{t+1}) = g^i(\theta^i_t, s_t, s_{t+1})$.

Example 2 *Consider an economy which consists of two consumers indexed* $i = 1, 2$*. The endowments to consumers are given by* $y^1_t = s_t$ *and* $y^2_t = 1$ *where* s_t *is a random variable governed by a two-state Markov chain with values* $s_t = s_1 = 0$*, and* $s_t = s_2 = 1$ *and transition matrix*

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}.$$

Also assume that household i orders consumption streams according to

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \ln[c^i_t(s^t)] \pi(s^t).$$

with $\beta = 0.95$. A recursive competitive equilibrium with sequential trading and Arrow securities is defined as an initial wealth distribution Θ_0, a pricing kernel $Q(s_{t+1} \mid s_t)$, value functions $\left\{v^i(\theta_t^i, s_t)\right\}_{i=1}^2$ and decision rules $\left\{h^i(\theta_t^i, s_t), g^i(\theta_t^i, s_t, s_{t+1})\right\}_{i=1}^2$ such that for $i \in \{1, 2\}$, given the initial wealth distribution and pricing kernel the decision rules solve the household problem and for all realizations of $\{s_t\}_{t=0}^{\infty}$, the consumption and asset portfolios implied by the decision rules satisfy $\sum_i c_t^i = \sum_i y^i(s_t)$ and $\sum_i \theta_{t+1}^i(s_{t+1}) = 0$ for all t and s_t. A natural borrowing limit for household i at state s_t is given by

$$A_t^i(s_t) = \sum_{\tau=t}^{\infty} \sum_{\{\widetilde{s}^\tau : \widetilde{s}^t = s^t\}} q_\tau^t(\widetilde{s}^\tau) y^i(\widetilde{s}_\tau).$$

For this case note that since $s_t \in \{0, 1\}$ there are only two Arrow securities (Q^1 and Q^2) traded at a given t one for each possible realization of s_t. Let's compute the prices of these Arrow securities. We need to compute four prices; two for each security for two possible states. Recall that the equilibrium prices must satisfy (15), therefore

$$\begin{aligned} Q_1^1 &= (0.95)\frac{h^i(\theta_t^i, 0)}{h^i(\theta_t^i, 0)}(0.8) \quad Q_2^1 = (0.95)\frac{h^i(\theta_t^i, 1)}{h^i(\theta_t^i, 0)}(0.3) \\ Q_1^2 &= (0.95)\frac{h^i(\theta_t^i, 0)}{h^i(\theta_t^i, 1)}(0.2) \quad Q_2^2 = (0.95)\frac{h^i(\theta_t^i, 1)}{h^i(\theta_t^i, 1)}(0.7) \end{aligned}$$

where Q_i^j denotes the price of the j^{th} security in the i^{th} state. We also reaffirm the previous result that consumption levels for households 1 and 2 must be perfectly correlated. Using the market clearing condition $\sum_i c_t^i(s^t) = \sum_i y^i(s_t)\ \forall s^t$ we find

$$\frac{h^i(\theta_t^i, 0)}{h^i(\theta_t^i, 1)} = \frac{1}{2} \text{ for } i = 1, 2.$$

which implies

$$\begin{aligned} Q_1^1 &= 0.76, \quad Q_2^1 = 0.57 \\ Q_1^2 &= 0.095, \quad Q_2^2 = 0.665. \end{aligned}$$

Note that we do not need to compute recursive competitive equilibrium allocations to figure out the state-contingent one-period-ahead claim prices. Using these prices we can, for instance, compute the price of a riskless asset which promises to deliver one unit of consumption good next period regardless of the realization of history. The price of this asset must be given by

$$\sum_{s_{t+1}} Q(s_{t+1} \mid s_t).$$

For the considered special case, the price of this asset when $s_t = 0$ must be $Q_1^1 + Q_1^2 = 0.76 + 0.095 = 0.855$ and $Q_2^1 + Q_2^2 = 0.57 + 0.665 = 1.235$ when $s_t = 1$.