1 Deterministic and Stochastic Growth

In what follows we consider the resource allocation in an economy composed of many identical, infinitely lived households. These households seek to maximize a discounted sum of utilities which depends on current and future consumption of a homogenous good. More specifically, the objective function, common to all households, is given by

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \tag{1}$$

where $0 < \beta < 1$. Output is produced using two inputs: Capital, k_t, and labor, n_t. It is assumed that labor is supplied inelastically, that is $n_t = 1$. A production function relates output to inputs, $y_t = F(k_t, n_t)$. Capital is assumed to depreciate at a constant rate, $0 < \delta \leq 1$. These assumptions introduce the following constraints to the maximization problem:

$$k_{t+1} = (1-\delta)k_t + i_t \tag{2}$$

$$c_t + i_t \leq y_t = F(k_t, n_t) \tag{3}$$

Therefore, the household problem can be defined as maximization of (1) subject to (2) and (3) given an initial capital stock, k_0.

1.1 The Deterministic Case

In this section we study the case without uncertainty, or in other words, the case with *perfect foresight*. Before we proceed to the analysis, a number of assumptions are in order:

i) The production function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is strictly increasing, homogenous of degree one, strictly quasi-concave and continuously differentiable with $F(0, n) = 0$, $F_k(k, n) > 0$, $F_n(k, n) > 0 \ \forall \ k, n$.

ii) $\lim\limits_{k \to 0} F_k(k, 1) = \infty$, $\lim\limits_{k \to \infty} F_k(k, 1) = 0$.

iii) The utility function is additively separable, that is $u(c_0, c_1, ...) = \sum\limits_{t=0}^{\infty} \beta^t U(c_t)$ and $\lim\limits_{c \to 0} U'(c) = \infty$.

Given these assumptions, we would like to consider, in the next step, the problem faced by a benevolent social planner, one whose objective is to maximize (1) by choosing the sequences $\{(c_t, k_{t+1}, n_t)\}_{t=0}^{\infty}$, subject to (2) and (3) given an initial capital stock, k_0. Note that (2) can be plugged into (3) to get rid of c_t. If we define $f(k) = F(k, 1) + (1 - \delta)k$, then the maximization problem now can be written as[1]:

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U[f(k_t) - k_{t+1}] \tag{4}$$

subject to

$$\begin{aligned} &0 \leq k_{t+1} \leq f(k_t), \quad t = 0, 1... \\ &k_0 > 0 \quad \text{given.} \end{aligned} \tag{5}$$

1.1.1 Finite Horizon

Even though the problem above is formulated for infinite horizon, it will be instructive to solve first the (easier) finite horizon case. Suppose that the planning horizon of the benevolent social planner is given by a finite value T. Note that with a finite horizon, the set of sequences $\{k_{t+1}\}_{t=0}^{T}$ satisfying (5) is a closed, bounded and convex subset of $\mathbb{R}^{T+1}$. Furthermore, by assumption, the objective function (4) is continuous and strictly concave.Therefore, the existance of a unique solution is ensured and can be characterized by the Kuhn-Tucker conditions. We can re-express (5) in the form of two distinct inequality constraints, $k_{t+1} \geq 0$ and $f(k_t) - k_{t+1} \geq 0$. Then, one can construct the Lagrangean function as follows:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t U[f(k_t) - k_{t+1}] + \sum_{t=0}^{T} \lambda_t (f(k_t) - k_{t+1}) + \sum_{t=0}^{T} \mu_t k_{t+1} \tag{6}$$

The Kuhn-Tucker conditions state that, if the sequence $\{k_{t+1}\}_{t=0}^{T}$ is a local maximum of the problem then there exist unique Lagrange multiplier sequences $\{\lambda_t\}_{t=0}^{T}$ and $\{\mu_t\}_{t=0}^{T}$ such that

$$\beta^{t+1} U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) - \beta^t U'(f(k_t) - k_{t+1}) - \lambda_t + \lambda_{t+1} f'(k_{t+1}) + \mu_t = 0 \tag{7}$$

[1]Note (and verify) that the assumptions on F imply f is continuously differentiable, strcitly increasing and strictly quasi-concave with $f(0) = 0$, $f'(k) > 0$, $\lim_{k \to 0} f'(k) = \infty$, $\lim_{k \to \infty} f'(k) = 1 - \delta$.

$$\lambda_t(f(k_t) - k_{t+1}) = 0 \text{ and } \lambda_t \geq 0, \tag{8}$$

$$\mu_t k_{t+1} = 0 \text{ and } \mu_t \geq 0. \tag{9}$$

Note that since $f(0) = 0$ and $U'(0) = \infty$, the inequality constraint, $f(k_t) - k_{t+1} \geq 0$ does not bind and it must also be the case that $k_{t+1} > 0$ for $0 \leq t \leq T-1$. Therefore, (8) and (9) imply that $\lambda_t = \mu_t = 0$ for $0 \leq t \leq T-1$. It then follows from (7) that

$$\beta U'(f(k_t) - k_{t+1}) f'(k_t) = U'(f(k_{t-1}) - k_t), \qquad t = 1, 2....T. \tag{10}$$

It is also clear from $f(0) = 0$ and $U'(0) = \infty$ and (7)-(9) that

$$k_{T+1} = 0 \tag{11}$$

Equation (10) implies a second-order difference equation. Among the set of solutions that satisfy (10) we are interested in the one that also satisfies the condition (11) and the initial value $k_0 > 0$. The following example is a good illustration of the solution procedure:

Example 1 *Suppose that $f(k) = k^\alpha$ with $0 < \alpha < 1$ and $U(c) = \ln c$. Let's solve for the equilibrium path for capital*[2]*. Verify that using the change of variable $z_t = k_t / k_{t-1}^\alpha$, the solution for z_t can be found as*

$$z_t = \alpha\beta \frac{1 - (\alpha\beta)^{T-t+1}}{1 - (\alpha\beta)^{T-t+2}}, \quad t = 1, 2, ..., T+1$$

therefore,

$$k_{t+1} = \alpha\beta \frac{1 - (\alpha\beta)^{T-t+1}}{1 - (\alpha\beta)^{T-t+2}} k_t^\alpha, \quad t = 0, 1, ..., T \tag{12}$$

1.1.2 Infinite Horizon

One can find the solution for the infinite horizon problem by taking the limit of the finite horizon solution allowing the planning horizon, T, to approach infinity. In fact, taking the limit in (12) we can find the infinite horizon solution as

[2] Do we violate, in this example, any assumptions we made in the beginning?

$$
\begin{aligned}
k_{t+1} &= \lim_{T\to\infty} \alpha\beta \frac{1-(\alpha\beta)^{T-t+1}}{1-(\alpha\beta)^{T-t+2}} k_t^{\alpha} \\
&= \alpha\beta k_t^{\alpha}.
\end{aligned} \tag{13}
$$

Even though this turns out to be an accurate conjecture, in this section we shall adopt a more general approach. Note that (13) describes an optimal saving decision based on current level of capital, k_t. In the infinite horizon case, the planning problem takes the same form every period, with only the beginning-of-period capital stock changing from one period to the next. Therefore, it is reasonable to conjecture that the optimal k_{t+1} can be defined only as a function of k_t. Suppose that problem (4)-(5) is solved for each possible value of k_0. Then we could define a function $v : \mathbb{R}_+ \to \mathbb{R}$ by taking $v(k_0)$ to be the value of maximized objective function (4), for each $k_0 \geq 0$. A function of this sort is called a *value function.* Then in terms of this value function the planner's problem in period-0 can be defined as:

$$
\begin{aligned}
&\max_{c_0,k_1} \left(U(c_0) + \beta v(k_1)\right) \\
&s.t.\ c_0 + k_1 \leq f(k_0) \\
&c_0, k_1 \geq 0 \\
&k_0 > 0 \quad \text{given}
\end{aligned} \tag{14}
$$

Note that we defined the value function as the maximized objective function (4) subject to (5) for a given level of initial capital stock. Therefore, v must satisfy

$$
v(k_0) = \max_{0\leq k_1 \leq f(k_0)} \left\{U(f(k_0) - k_1) + \beta v(k_1)\right\}
$$

where we plugged the constraint in the objective function. When we define the objective function in such a *recursive* way, the time subscript becomes a nuisance, that is, the date is not important. We face the same problem as we move along the planning horizon with changing levels of initial capital stock. One can rewrite the problem facing a planner with current capital stock, k, as

$$
v(k) = \max_{0\leq y \leq f(k)} \left\{U(f(k) - y) + \beta v(y)\right\}. \tag{15}
$$

Keep in mind that we do not know the functional form of v. Equation (15) is called a *functional equation.* The study of dynamic optimization problems through the analysis of such functional equations is called *dynamic programming*. If we knew that the function v was differentiable and the maximizing value of y (call it $g(k)$) was interior, then the first-order and envelope conditions for (15) would give

$$U'(f(k) - g(k)) = \beta v'(g(k)),$$

$$v'(k) = f'(k)U'\left(f(k) - g(k)\right),$$

respectively. The first equation equates the marginal utility of consuming current output to the marginal utility of allocating it to capital and enjoying augmented consumption next period. The second condition states that the marginal value of current capital, in terms of total discounted utility, is given by the marginal utility of using the capital in current production and allocating its return to current consumption.

1.1.3 The Contraction Mapping Theorem

In this section, we introduce a powerful theorem which proves the existence and uniqueness of solutions in certain dynamic programming problems of interest. First, let's define the concepts of metric space and contraction mapping.

Definition 1 *A metric space is a set S, together with a metric (distance function)* $\rho : S \times S \rightarrow \mathbb{R}$, *such that for all* x, y, $z \in S$:

a. $\rho(x, y) \geq 0$, *with equality if and only if* $x = y$;

b. $\rho(x, y) = \rho(y, x)$; *and*

c. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

For instance, (verify that) the set of integers with $\rho(x, y) = |x - y|$ is a metric space. Or the set of all continuous, strictly increasing functions on $[a, b]$, with $\rho(x, y) = \max_{a \leq t \leq b} |x(t) - y(t)|$ is also a metric space.

Definition 2 *Let (S,ρ) be a metric space and* $T : S \rightarrow S$ *be a function mapping S into itself.* T *is a contraction mapping with modulus* β *if for some* $\beta \in (0, 1)$, $\rho(Tx, Ty) \leq \beta\rho(x, y)$, *for all* x, $y \in S$.

The *fixed points* of a contraction mapping T are defined as the elements of S satisfying $Tx = x$.

Theorem 1 *(Contraction Mapping Theorem) If (S,ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus β, then*

a. *T has exactly one fixed point v in S, and*

b. *for any $v_0 \in S$, $\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$, $n = 0, 1, 2....$*

One can verify that certain operators are contractions using Blackwell's sufficiency condition.

Theorem 2 *(Blackwell's sufficient conditions for a contraction) Let X$\subseteq R^l$ and let B(X) be a space of bounded functions f:X$\to$ R, with the sup metric. Let T:B(X)$\to$ $B(X)$ be an operator satisfying*

1. *(monotonicity) f,g $\in B(X)$ and $f(x) \leq g(x)$, for all $x \in X$, implies $(Tf)(x) \leq (Tg)(x)$, for all $x \in X$;*
2. *(discounting) there exists some $\beta \in (0,1)$ such that $(T(f+a))(x) \leq (Tf)(x) + \beta a$, all $f \in B(X)$, $a \geq 0$, $x \in X$.*

Then T is a contraction mapping with modulus β.

We can employ Blackwell's sufficient conditions to show that the functional equation (15) represents a contraction. It is quite easy to verify that the mapping

$$(Tv)(k) = \max_{0 \leq y \leq f(k)} \{U(f(k) - y) + \beta v(y)\}$$

satisfies both of the Blackwell conditions. If $v(y) \leq \varpi(y)$ $\forall y$, then the objective function for which $T\varpi$ is the maximized value is uniformly higher than the function for which Tv is the maximized value; so the monotonicity is obvious. The discounting property can also be verified as follows:

$$\begin{aligned} T(v+a)(k) &= \max_{0 \leq y \leq f(k)} \{U(f(k) - y) + \beta(v(y) + a)\} \\ &= \max_{0 \leq y \leq f(k)} \{U(f(k) - y) + \beta v(y)\} + \beta a \\ &= (Tv)(k) + \beta a. \end{aligned}$$

Hence, we have shown that the mapping defined by (15) has a unique fixed point. Now, given this information, one can use contraction properties to find that fixed point, in other words, the unknown value function $v(k)$. Note that the fact that (15) is a contraction implies the existance of a time-invariant value function. It follows from the contraction mapping theorem that for any $v_0(k)$, $\rho(Tv_0, Tv) = \rho(Tv_0, v) \leq \beta\rho(v_0, v)$. This means that if we start from an arbitrary guess about the value function and apply the mapping, the resulting function will be closer to the real value function which satisfies (15). The following example shows how this type of *successive approximations* can be used to find the value function.

Example 2 *Suppose, once again, that $f(k) = k^\alpha$ with $0 < \alpha < 1$ and $U(c) = \ln c$. Then the functional equation can be written as:*

$$v(k_t) = \max_{0 \leq k_{t+1} \leq f(k_t)} \{\ln(k_t^\alpha - k_{t+1}) + \beta v(k_{t+1})\} \tag{16}$$

Let's conjecture that the value function is just equal to a constant, that is $v(k_t) = c > 0$. If this is a true conjecture, then $T(ak_t)$ must also be of the same functional form. Working on (16) it can be found that

$$T(ak_t) = \alpha \ln(k_t) + \beta c$$

which is obviously a different functional form. Yet, we know that we got closer to the true value function. At this point, tipped by the outcome of the first iteration, we can actually make a more "educated" guess. Since the value function appears to be logarithmic, let's guess and verify that it is of the form $A \ln(k_t) + B$. The first-order condition implied by (16) gives the optimal saving decision as

$$k_{t+1} = \frac{\beta A}{\beta A + 1} k_t^\alpha. \tag{17}$$

Plugging this expression back into the functional equation gives

$$v(k_t) = (1 + \beta A) \ln k_t^\alpha + \beta A \ln \frac{\beta A}{1 + \beta A} + \ln \frac{1}{1 + \beta A} + \beta B. \tag{18}$$

Equation (18) verifies our conjecture and it can be found that

$$A = \frac{\alpha}{1 - \alpha\beta}$$

$$B = \frac{1}{1 - \beta} \left(\beta A \ln \frac{\beta A}{1 + \beta A} + \ln \frac{1}{1 + \beta A} \right)$$

Now with the knowledge of the value function and the parameter A equation (17) can be used to find the time-invariant saving decision rule as

$$k_{t+1} = \alpha\beta k_t^{\alpha}. \tag{19}$$

This confirms the validity of taking the limit of the finite horizon solution to find the solution for the infinite horizon case. Furthermore, using this result, the steady-state capital stock, k^*, can be found as

$$k^* = (\alpha\beta)^{\frac{1}{1-\alpha}}$$

Not so surprisingly, steady-state capital increases as households become more patient and capital productivity (measured by the parameter α) improves.

1.1.4 Linearization Approach

Most of the problems we face unfortunately do not have analytical solutions. The difficulty mainly stems from the non-linearity of first-order optimality conditions as in (10). In such cases, rather than working on the original non-linear equations, solving an approximate linearized system of equations may prove much easier. This approach involves the use of Taylor's theorem and the implied approximation method. The following example demonstrates, in steps, how we can use this approach to solve the planner's problem in the economy considered above.

Suppose that $U(c_t) = \ln c_t$ and $f(k_t) = k_t^{\alpha}$. A Lagrangean can be formed in the following way to solve the planner's maximization problem:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \ln c_t + \lambda_t (k_t^{\alpha} - c_t - k_{t+1}) \right\}$$

From our previous experience we already know that the optimal comsumption and saving must be positive for all t and the constraint must bind. Together with these prior results, first-order conditions give:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t} &= \beta^t \frac{1}{c_t} - \beta^t \lambda_t = 0 \\ \frac{\partial \mathcal{L}}{\partial k_{t+1}} &= \beta^t \lambda_t - \beta^{t+1} \alpha k_{t+1}^{\alpha-1} \lambda_{t+1} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_t} &= k_t^{\alpha} - c_t - k_{t+1} = 0 \end{aligned} \tag{20}$$

Note that (20) describes a system of first-order non-linear difference equations with three variables: c, k and λ.We can work on (20) and obtain the following Euler equation by combining the first two equations of the system.

$$\frac{1}{c_t} = \frac{\alpha\beta k_{t+1}^{\alpha-1}}{c_{t+1}}$$

By doing so we simplify (20) and obtain the following two-variable non-linear system with two equations:

$$\begin{aligned} k_{t+1} &= k_t^{\alpha} - c_t \\ \frac{\alpha\beta k_{t+1}^{\alpha-1}}{c_{t+1}} &= \frac{1}{c_t} \end{aligned} \tag{21}$$

It is not possible to find an explicit analytical solution for (21). At this point, instead of working on (21) we may rather work on an approximate system of linear difference equations. In fact, this approach is quite practical and useful and has a wide range of economic applications. However, before going into the details of how we go about approximating the system let's remember an important mathematical result.

Theorem 3 *(Taylor's Theorem) Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ ((n-1)th derivative of the function f) is continious on $[a, b]$, $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define*

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

Then there exists a point x between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n.$$

In general, Taylor's theorem shows that a function f can be approximated by a polinomial of degree $(n-1)$ by expanding it around a point x_0 as follows:

$$f(x) \approx f(x_0) + f^1(x_0)(x-x_0) + \frac{f^2(x_0)}{2}(x-x_0)^2 + ... \tag{22}$$

We can, therefore, Taylor-expand the equations of the system (21) to obtain a system of linear difference equations. In other words, we can linearize (21) around the steady-state and solve the resulting system of equations to find the optimal consumption and saving decisions at time t so that the system will eventually converge to the steady-state. The steady-state point can easily be found by solving

$$\begin{aligned} k &= k^{\alpha} - c \\ \alpha\beta k^{\alpha-1} &= 1 \end{aligned}$$

which gives $k = (\beta\alpha)^{\frac{1}{1-\alpha}}$ and $c = (1-\beta\alpha)(\beta\alpha)^{\frac{\alpha}{1-\alpha}}$. Now we can Taylor approximate (21) around this steady state as follows:

$$\begin{aligned} k_{t+1} &= k + (\alpha k^{\alpha-1})(k_t - k) - (c_t - c) \\ c_t &= c + \left(\frac{1}{\alpha\beta k^{\alpha-1}}\right)(c_{t+1} - c) + \left((1-\alpha)\frac{ck^{-\alpha}}{\beta\alpha}\right)(k_{t+1} - k) \end{aligned} \tag{23}$$

Note that now we can express both capital and consumption as functions of absolute deviations of the same variables from the steady-state values. In most applications, it makes sense to evaluate the problem by percentage change rather than absolute change. Let's define

$$\widetilde{x}_t = \frac{x_t - x}{x} \approx \ln x_t - \ln x. \tag{24}$$

Note that approximation (24) holds quite well for small deviations around the steady-state. Now given (24), we can rewrite (23) in terms of percentage deviations as follows:

$$\begin{aligned} \widetilde{k}_{t+1} &= (\alpha k^{\alpha-1})\widetilde{k}_t - \left(\frac{c}{k}\right)\widetilde{c}_t \\ \widetilde{c}_t &= \left(\frac{1}{\alpha\beta k^{\alpha-1}}\right)\widetilde{c}_{t+1} + \left(\frac{(1-\alpha)k^{1-\alpha}}{\beta\alpha}\right)\widetilde{k}_{t+1} \end{aligned} \tag{25}$$

After plugging in the values for k and c, (25) reduces to

$$\begin{aligned} \widetilde{c}_{t+1} + (1-\alpha)\widetilde{k}_{t+1} &= \widetilde{c}_t \\ -(\beta\alpha)\widetilde{k}_{t+1} &= (1-\beta\alpha)\widetilde{c}_t - \alpha\widetilde{k}_t. \end{aligned} \tag{26}$$

The system of equations (26) canbe examined compactly in the matrix form:

$$\begin{bmatrix} 1 & 1-\alpha \\ 0 & -\alpha\beta \end{bmatrix} \begin{bmatrix} \widetilde{c}_{t+1} \\ \widetilde{k}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1-\alpha\beta & -\alpha \end{bmatrix} \begin{bmatrix} \widetilde{c}_{t} \\ \widetilde{k}_{t} \end{bmatrix} \tag{27}$$

Inverting the matrix on the left hand side the system becomes:

$$\begin{bmatrix} \widetilde{c}_{t+1} \\ \widetilde{k}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1-\alpha+\beta\alpha^2}{\beta\alpha} & \frac{\alpha-1}{\beta} \\ \frac{\beta\alpha-1}{\beta\alpha} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \widetilde{c}_{t} \\ \widetilde{k}_{t} \end{bmatrix} \tag{28}$$

We can obtain a solution for the system of difference equations given in (28) by applying Jordan decomposition. A square matrix M can be Jordan-decomposed to obtain $M = ABA^{-1}$ where B is a dioagonal matrix whose diagonal elements are the eigenvalues of M and A is a matrix of the corresponding eigenvectors. The matrix on the right hand side of (28) can be decomposed to get

$$A^{-1} = \frac{1}{1-\beta\alpha^2} \begin{bmatrix} 1-\beta\alpha & 1-\alpha \\ 1 & -\alpha \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha\beta} \end{bmatrix}.$$

So the system can be rewritten as:

$$\begin{bmatrix} 1-\beta\alpha & 1-\alpha \\ 1 & -\alpha \end{bmatrix} \begin{bmatrix} \widetilde{c}_{t+1} \\ \widetilde{k}_{t+1} \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha\beta} \end{bmatrix} \begin{bmatrix} 1-\beta\alpha & 1-\alpha \\ 1 & -\alpha \end{bmatrix} \begin{bmatrix} \widetilde{c}_{t} \\ \widetilde{k}_{t} \end{bmatrix}$$

More explicitly, the equations can be written as:

$$\begin{aligned} (1-\alpha\beta)\widetilde{c}_{t+1} + (1-\alpha)\widetilde{k}_{t+1} &= \alpha\left((1-\alpha\beta)\widetilde{c}_{t} + (1-\alpha)\widetilde{k}_{t}\right) \\ \widetilde{c}_{t+1} - \alpha\widetilde{k}_{t+1} &= \frac{1}{\alpha\beta}\left(\widetilde{c}_{t} - \alpha\widetilde{k}_{t}\right) \end{aligned}$$

These are simple first-order difference equations and, by backward iteration, the second equation can be written as:

$$\widetilde{c}_{t} - \alpha\widetilde{k}_{t} = \left(\frac{1}{\alpha\beta}\right)^{t}\left(\widetilde{c}_{0} - \alpha\widetilde{k}_{0}\right)$$

Since $\frac{1}{\alpha\beta} > 1$, $\widetilde{c}_t - \alpha\widetilde{k}_t$ is diverging at the rate $\frac{1}{\alpha\beta}$. Therefore, this equation implies an explosive path for a certain linear combination of consumption and capital percentage deviations (from the steady-state). This possibility must be rules out as it is seldom optimal and do not bring usa steady-state which we are after. Therefore it must be the case that

$$\widetilde{c}_t = \alpha \widetilde{k}_t.$$

Note that the above equation gives us an optimal decision rule in terms of percentage deviations. Compare this result to the decision rule we found (in levels) using dynamic programming and verify that it is nothing but a linearized version of the optimal consumption decision rule implied by (19).

1.1.5 The Method of Undetermined Coefficients

Recall that we defined the optimal policy function in the previous part as a mapping from the state to the choice variables. This implies that we can actually guess the form of the relationship between $\widetilde{c}_t$ and $\widetilde{k}_t$. Since the system is linear, it is appropriate to conjecture that $\widetilde{c}_t = \varepsilon \widetilde{k}_t$. Our goal is to find the unknown coefficient ε. Let's reconsider the system (27) together with our conjecture.

$$\begin{bmatrix} 1 & 1-\alpha \\ 0 & -\alpha\beta \end{bmatrix} \begin{bmatrix} \varepsilon \widetilde{k}_{t+1} \\ \widetilde{k}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1-\alpha\beta & -\alpha \end{bmatrix} \begin{bmatrix} \varepsilon \widetilde{k}_t \\ \widetilde{k}_t \end{bmatrix} \tag{29}$$

Writing down the equations explicitly we have

$$(\varepsilon + 1 - \alpha)\varepsilon \widetilde{k}_{t+1} = \varepsilon \widetilde{k}_t \tag{30}$$

and

$$-\alpha\beta \widetilde{k}_{t+1} = [(1-\alpha\beta)\varepsilon - \alpha]\widetilde{k}_t \tag{31}$$

When we divide (30) by (31) we get

$$\frac{\varepsilon + 1 - \alpha}{-\alpha\beta} = \frac{\varepsilon}{(1-\beta\alpha)\varepsilon - \alpha}.$$

Rearranging the terms we obtain

$$(\varepsilon - \alpha)[(1-\beta\alpha)\varepsilon + 1 - \alpha] = 0.$$

This equation has two roots which can be found as:

$$
\begin{aligned}
\varepsilon_1 &= \alpha \\
\varepsilon_2 &= \frac{-(1-\alpha)}{1-\beta\alpha}
\end{aligned}
$$

For the first root we have $\alpha\widetilde{k}_t = \widetilde{c}_t$. Since $\alpha < 1$ this is a stable first-order difference equation. For the second root, however, we get $\widetilde{k}_{t+1} = (\beta\alpha)^{-1}\widetilde{k}_t$. Since $\alpha\beta < 1$ this root implies an explosive first-order difference equation. Therefore, this cannot be the solution. Note that the solution implied by the first root is also the solution we found using the Jordan decomposition method.

1.2 The Stochastic Case

In this section we start taking into account uncertainty which effects technology in a specific way. Assume that output is given by

$$y_t = a_t f(k_t)$$

where $\{a_t\}$ is a sequence of independently and identically distributed (i.i.d) random variables and f is defined as in the previous section. Thus, the constraint the planner faces becomes

$$k_{t+1} + c_t \leq a_t f(k_t), \qquad c_t,\ k_{t+1} \geq 0 \quad \forall t, \{a_t\}$$

Preferences now can be expressed in the expected value form, that is, households now rank consumption streams with respect to the utility they are expected to deliver. The utility function takes the same additively separable form:

$$E[u(c_0, c_1, ...)] = E_0 \sum_{t=0}^{\infty} \beta^t U(c_t)$$

where $E_0[.]$, denotes the expectations operator given time-0 information. We assume that at the beginning of period t the current value a_t is realized. Thus, the pair (k_t, a_t) and the value of total output $a_t f(k_t)$ are known when consumption c_t takes place and end-of-period capital k_{t+1} is accumulated. The pair (k_t, a_t) is called the *state* of the economy at date t. The maximum amount of utility that can be obtained by starting the program with (k_t, a_t) is given by $v(k_t, a_t)$. Therefore, we can construct the value function as follows:

$$v(k_t, a_t) = \max_{0 \le k_{t+1} \le f(k_t)} \{U(a_t f(k_t) - k_{t+1}) + \beta E[v(k_{t+1}, a_{t+1}) \mid \Omega_t]\}$$

where Ω_t denotes the information set at time t.The methods used to characterize the optimal policy in the stochastic case are analogous to those used for the deterministic case. Thus, assuming that the value function is differentiable and the solution is interior, the first-order conditions give

$$U'(a_t f(k_t) - k_{t+1}) = \beta E_t[v_1(k_{t+1}, a_{t+1})].$$

Note that the above equation gives an optimal decision rule of the form $k_{t+1} = g(k_t, a_t)$. One can put this expression back into the functional equation, take the derivative with respect to k_t and use the envelope theorem to obtain the stochastic analogy of (10):

$$U'(a_t f(k_t) - k_{t+1}) = \beta E_t[a_{t+1} f'(k_{t+1}) U'(a_{t+1} f(k_{t+1}) - k_{t+2})]$$

The difference of the functional equation in the stochastic case is the expectation operator that appear on the right-hand-side. However note that the mapping, $\max\{U[.] + \beta E_t[v(.,.)]\}$, is still a contraction since it satisfies Blackwell sufficiency. This means that we can still apply the successive approximations method to find the value function. If we apply the same guess-verify method the implied decision rule can be found as

$$k_{t+1} = \alpha\beta(a_t k_t^\alpha). \tag{32}$$

Note that (32) is a stochastic version of (17). Now suppose that the cumulative distribution function of the random variable a_t is denoted by $G(.)$. Given the initial capital stock k_0, next period's capital stock's distribution function, ψ_1, is determined by $G(.)$ such that

$$\begin{aligned} \psi_1(a) &= \Pr(k_1 \le a) = \Pr(\alpha\beta a_0 k_0^\alpha \le a) \\ &= \Pr(a_0 \le a/\alpha\beta k_0^\alpha) = G(a/\alpha\beta k_0^\alpha). \end{aligned}$$

Since the same logic holds for any successive pair of periods, we can define the function

$$H(a, b) = \Pr(k_{t+1} \le a \mid k_t = b) = G(a/\alpha\beta b^\alpha), \quad a,\ b > 0.$$

H is called a *transition function.* Given this definition, the sequence of unconditional distribution functions $\{\psi_t(.)\}_{t=1}^{\infty}$ can be defined as

$$\psi_{t+1}(a) = \Pr(k_{t+1} \leq a) = \int H(a,b) d\psi_t(b), \quad t = 0, 1, 2, ...$$

If g and G are in some suitable families, then this sequence converges to a limiting distribution function ψ satisfying

$$\psi(k_{t+1}) = \int H(a,b) d\psi(k_t). \tag{33}$$

The time-invariant function ψ given by (33) is called an *invariant distribution.* An invariant distribution is a stochastic analogue to a stationary point of a deterministic system.

1.2.1 Discrete-State Numerical Dynamic Programming

In this section, we present an example of how we can solve Bellman equations numerically for discrete state-space problems. The stochastic growth problem we have been studying so far, in fact, is not a discrete state-space problem. However, for the time being, suppose that the state-space is discrete such that there are n states $[k_1, k_2, ..k_n]$ for capital and two states $[a_1, a_2]$ for the productivity parameter. Also assume that productivity parameter follows a discrete Markov process with a transition matrix P. Define two $n \times 1$ vectors v_j, $j = 1, 2$ whose i^{th} rows are determined by $v_j(i) = v(k_i, a_j)$. Let $\mathbf{1}$ be the $n \times 1$ matrix consisting entirely of ones. Define two $n \times n$ matrices R_j whose (i, h) element is given by

$$R_j(i,h) = U[a_j f(k_i) - k_h], \; i = 1, 2...n, \; h = 1, 2....n$$

Define an operator $T([v_1, v_2])$ that maps a pair of vectors $[v_1, v_2]$ into a pair of vectors $[tv_1, tv_2]$:

$$\begin{aligned} tv_1 &= \max\{R_1 + \beta P_{11}\mathbf{1}v_1' + \beta P_{12}\mathbf{1}v_2'\} \\ tv_2 &= \max\{R_2 + \beta P_{21}\mathbf{1}v_1' + \beta P_{22}\mathbf{1}v_2'\} \end{aligned} \tag{34}$$

More explicitly, the matrix on the right hand side of the first equation is of the following form:

$$\left[\begin{array}{llll} U[a_1 f(k_1) - k_1] + \beta(P_{11} v_1(k_1) + P_{12} v_2(k_1)) & . & . & U[a_1 f(k_1) - k_n] + \beta(.) \\ U[a_1 f(k_2) - k_1] + \beta(P_{11} v_1(k_1) + P_{12} v_2(k_1)) & . & . & U[a_1 f(k_2) - k_n] + \beta(.) \\ \vdots & \vdots & \vdots & \vdots \\ U[a_1 f(k_n) - k_1] + \beta(P_{11} v_1(k_1) + P_{12} v_2(k_1)) & . & . & U[a_1 f(k_n) - k_n] + \beta(.) \end{array}\right]$$

Here the "max" operator applied to an $n \times m$ matrix M returns an $n \times 1$ vector whose i^{th} row is the maximum element of the i^{th} row of the matrix M[3]. We can write (34) more compactly in the following way:

$$\left[\begin{array}{c} tv_1 \\ tv_2 \end{array}\right] = \max \left\{ \left[\begin{array}{c} R_1 \\ R_2 \end{array}\right] + \beta(P \otimes \mathbf{1}) \left[\begin{array}{c} v_1' \\ v_2' \end{array}\right] \right\}$$

The Bellman equation can now be written as $[v_1, v_2] = T([v_1, v_2])$ and can be solved by iterating until the following sequence converges:

$$[v_1, v_2]_{m+1} = T([v_1, v_2]_m)$$

1.3 Competitive Equilibrium

So far, we have considered the problem faced by a hypothetical social planner. In what follows, we will show that the optimal allocations implied by the social planner's problem can actually be interpreted as market outcomes under certain conditions. In other words, we will decentralize the economy we have been discussing so far by introducing input and output markets. First, we have to specify the structure of markets as well as the ownership rights of the consumers. Assume that households own all factors of production and all shares in firms and that these endowments are equally distributed across households. Also assume that firms hire labor and capital on a rental basis to produce output each period and return any profits that result to shareholders.

Let p, w and r denote the price of a unit of output, the real wage and the real rental price of capital respectively. Also assume that all trading takes place in the period 0. No further trades are negotiated later. Also for simplicity let the planning horizon be finite and given by T. Given this setting, firms' problem can be summarized as choosing input demands and output supplies $\{k_t, n_t, y_t\}_{t=0}^T$, given the prices $\{p_t, r_t, w_t\}_{t=0}^T$ to solve the following problem:

[3] For an $n \times m$ matrix A, the Matlab command [r,index]=max(A) returns two $1 \times n$ row vectors r, index where $r_j = \max_i A(i,j)$ and index$_j$ is the row i that attains $\max_i A(i,j)$ for column j.

$$\max\ \pi = \sum_{t=0}^{T} p_t(y_t - r_t k_t - w_t n_t) \tag{35}$$
$$\text{subject to } \ y_t \leq F(k_t, n_t), \quad t = 0, 1, ...T$$

Given the same price sequences, the typical household must choose demand for consumption and investment, and supplies of current capital and labor, $\{c_t, i_t, x_{t+1}, k_t, n_t\}_{t=0}^{T}$ to solve the following problem:

$$\begin{aligned} \max \quad & \sum_{t=0}^{T} \beta^t U(c_t) \quad (36) \\ s.t. \quad & \sum_{t=0}^{T} p_t(c_t + i_t) \leq \sum_{t=0}^{T} p_t(r_t k_t + w_t n_t) + \pi \\ & x_{t+1} = (1-\delta)x_t + i_t, \quad t = 0, 1, ...T \\ & 0 \leq n_t \leq 1, \ 0 \leq k_t \leq x_t, \quad t = 0, 1, ...T \\ & c_t, x_{t+1} \geq 0, \quad t = 0, 1, ...T \end{aligned}$$

Now we can define a competitive equilibrium as follows: A competitive equilibrium is a set of prices $\{p_t, r_t, w_t\}_{t=0}^{T}$, an allocation $\left\{k_t^d, n_t^d, y_t\right\}_{t=0}^{T}$ for the typical firm, and an allocation $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{T}$ for the typical household, such that

- $\left\{k_t^d, n_t^d, y_t\right\}_{t=0}^{T}$ solves the firm problem given the price sequences $\{p_t, r_t, w_t\}_{t=0}^{T}$;
- $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{T}$ solves the household problem given $\{p_t, r_t, w_t\}_{t=0}^{T}$;
- all markets clear: $k_t^d = k_t^s$, $n_t^d = n_t^s$, $c_t + i_t = y_t$, $\forall t$.

In order to solve for the competitive equilibrium allocations we conjecture that $p_t > 0$ and $r_t, w_t > 0$. Given these, the firm supplies to the market all the goods it produces, that is $y_t = F(k_t, n_t)$. The solution of (35) gives the following:

$$\begin{aligned} r_t &= F_k(k_t, n_t) \quad (37) \\ w_t &= F_n(k_t, n_t) \end{aligned}$$

Since the production function is homogenous of degree one, substitution of (37) into (35) gives $\pi = 0$. Now let's evaluate the houshold problem. Since $r_t, w_t > 0$, we must have $n_t = 1$ and $k_t = x_t$. Given these facts we can now redefine the houshold problem as follows:

$$\begin{array}{ll} \max & \sum_{t=0}^{T} \beta^t U(c_t) \\ s.t. & \sum_{t=0}^{T} p_t \left(c_t + k_{t+1} - (1-\delta+r_t)k_t - w_t\right) \leq 0 \\ & c_t, k_{t+1} \geq 0 \text{ and } k_0 = x_0 \text{ is given.} \end{array}$$

We can proceed in a similar way as we did before to find the first-order conditions of this maximization problem. Given $\lim_{c \to 0} U'(c) = \infty$, the first-order conditions are given by

$$\begin{aligned} \beta^t U'(c_t) - \lambda p_t &= 0 \\ \lambda(p_t - (r_{t+1} + 1 - \delta)p_{t+1}) &\leq 0 \text{ with equality if } k_{t+1} > 0 \end{aligned} \tag{38}$$

where λ is the Lagrange multiplier associated with the budget constraint. Therefore, the competitive equilibrium is given by the sequences $\left\{c_t^e, k_{t+1}^e, p_t^e, r_t^e, w_t^e\right\}_{t=0}^{T}$ and initial value $k_0 = x_0$ which satisfy (37) and (38) together with the resource constraint

$$F(k_t^e, 1) = c_t^e + k_{t+1}^e - (1-\delta)k_t^e$$

and $k_{T+1} = 0$.

Note that the competitive equilibrium allocations are pareto-optimal. In other words, this set of allocations is the one which would be chosen by benevolent social planner. To see this, suppose thatthere exists another feasible allocation $\left\{c_t', k_{t+1}'\right\}_{t=0}^{T}$ which yields a higher level of utility for households. Then this allocation must violate the housheold budget constraint, otherwise the household would have chosen it. But if the housheold budget constraint is violated then we have

$$\pi' = \sum_{t=0}^{T} p_t^e \left(F(k_t', 1) - r_t k_t - w_t n_t\right) > 0 = \pi^e$$

contradicting the result that $\{k_t^e, n_t\}_{t=0}^{T}$ is a profit-maximizing choice of inputs.