2 First Steps in Symbolization

1 The sentential connectives

The premises and the conclusions of the arguments of Chapter One are complete sentences of English, but some of these sentences themselves consist in shorter complete sentences combined into a single sentence using ‘and’, ‘if… then…’ and so on. For example, premise 1 of A on page 3, ‘If our currency loses value then our trade deficit will narrow’, is a *compound* sentence whose *constituent* sentences are ‘our currency loses value’ and ‘our trade deficit will narrow’, connected together using ‘if…then…’. A phrase like ‘if…then…’, which can join shorter sentences to make a longer one, is called a *sentential connective*, sentential logic being that system of logic concerned with the logic of sentential connectives. To be more specific, in our presentation it is concerned with the logic of exactly five sentential connectives:

- it is not the case that…
- either…or…
- both…and…
- if…then…
- …if and only if…

In this list, each occurrence of an ellipsis ‘…’ indicates a position which may be filled by a complete sentence (which may itself contain further connectives). A connective which attaches to one sentence to form a compound sentence is called a *one-place* connective, and a connective which combines with two sentences to form a compound sentence is called *two-place*. So in our list, there is a single one-place connective, ‘it is not the case that…’, and the others are all two-place.

Here are some technical terms which we will often use in talking about sentences and sentential connectives:

- A sentence which contains no occurrence of any sentential connective is called a *simple* or *atomic* sentence.
- A sentence which contains at least one occurrence of at least one sentential connective is called a *complex* or *compound* sentence.

Writing out the connectives in English whenever they occur rapidly becomes tiresome: a shorthand notation saves space and time. We will therefore settle on the following abbreviations:

- For 'it is not the case that…' write '~…'
- For 'both…and…' write '… & …'
- For 'either…or…' write '… ∨ …'
- For 'if…then…' write '… → …'
- For '…if and only if…' write '… ↔ …'

There is important terminology which attaches to these abbreviations, and which should be memorized:

- A sentence beginning with '~' is a *negation*, and the subsentence to which the '~' is prefixed is the *negated sentence*.
- A sentence consisting in two shorter sentences connected by '&' is a *conjunction*, and the connected sentences are the *first* and *second conjuncts* respectively.
- A sentence consisting in two shorter sentences connected by '∨' is a *disjunction*, and the connected sentences are the *first* and *second disjuncts* respectively.
- A sentence consisting in two shorter sentences connected by '→' is a *conditional;* the shorter sentence before the '→' is the *antecedent* and the shorter sentence after the '→' is the *consequent*.
- A sentence consisting in two shorter sentences connected by '↔' is a *biconditional;* the shorter sentence before the '↔' is the *left-hand side* and the shorter sentence after the '↔' is the *right-hand side*.

The symbols themselves also have names, though these are less important. '~' is the familiar *tilde* symbol from typewriter or computer keyboards; '∨' is called the *wedge*, or sometimes the *vel* (after the Latin word 'vel' for 'or'); '&' is the *ampersand* symbol, again familiar from typewriter or computer keyboards; '→' is called the *arrow;* and '↔' is called the *double-arrow*. In other logic textbooks one sees different symbols for these five English sentential connectives, but we will use the ones just listed exclusively.

There are of course many other sentential connectives besides the five listed. For example, there is 'neither…nor…' as in 'neither will our currency lose value nor will our trade deficit narrow' (or more colloquially, 'our currency will not lose value, nor will our trade deficit narrow'), and there is '…after…', as in 'our currency will lose value after our trade deficit narrows'. These sentential connectives and others like them fall into two groups: in the first group there are connectives like 'neither…nor…' which are *definable* in terms of the five connectives we have already introduced, and in the second group there are connectives like 'after' which are, as we might put it, beyond the scope of classical sentential logic (what this means will be fully explained in §6 and §7 of Chapter 3). So in Part I of this book our five connectives '~', '&', '∨', '→' and '↔' are all we need.

2 Negations, conjunctions and disjunctions

The simplest sentences to formalize are those which contain no occurrences of connectives, that is, atomic sentences. To symbolize an atomic sentence one simply uses a single letter, known as a *sentence-letter*. Thus

(1) Our currency will lose value

is symbolized as, say,

(1.s) C.

Sentences with one explicitly employed connective are equally straightforward, as in

(2) Our currency will lose value and our trade deficit will narrow

which is

(2.s) C & N

where 'N' stands for 'our trade deficit will narrow'.

However, when two or more connectives occur in a sentence, problems of interpretation can arise. Consider

(3) Either our currency will lose value, or exports will decrease and inflation will rise.

Using 'E' for 'exports will decrease' and 'I' for 'inflation will rise', we could symbolize (3) as

(3*) C ∨ E & I.

But (3*) can be read two ways, as a conjunction whose first conjunct is 'C ∨ E' or as a disjunction whose second disjunct is 'E & I'. These correspond to two different ways of reading the English, but in English the correct interpretation is settled by the position of the comma: everything that follows the comma is grouped together. So (3) is a disjunction whose second disjunct is 'exports will decrease and inflation will rise'. In symbolizations we do not use commas, but we can get the same effect by using parentheses to group the appropriate sentence-letters together. The correct symbolization of (3) is written:

(3.s) C ∨ (E & I).

(3.s) is to be distinguished from (4), which is the incorrect reading of (3):

(4) (C ∨ E) & I.

The difference between (3.s) and (4) can be brought out by comparing the following two English sentences which instantiate them, 'either I will keep to the speed limit, or I will break the speed limit and I will be killed', versus 'either I will keep to the speed limit or I will break the speed limit, and I will be killed'. Only the second of these says that death is inevitable.

The difference between (3.s) and (4) introduces the idea of the *scope* of a connective in a sentence. We will later give a precise definition of the notion of scope, but for the moment let us say that the scope of a two-place connective in a sentence, or *formula*, is that part of the formula enclosed in the closest pair of matching parentheses within which the connective lies, if there is such a pair, or else the entire formula, if there is not. So the scope of '&' in (3.s) is 'E & I' while the scope of '&' in (4) is the entire formula. The function of the parentheses in (3.s) and (4) is to tell us what reading of (3*) is intended, but we can also describe this function as that of *fixing the scope* of the connectives in (3*), since the different ways of reading (3*) differ only in assigning different scopes to the two connectives which occur in each: in (3.s), '∨' has '&' within its scope, while in (4), '&' has '∨' within its scope. The *main* connective of a formula is that connective in it which is not within the scope of any other. So '∨' is the main connective of (3.s) and '&' is the main connective of (4). Again, we will give a more adequate definition later.

Parentheses are only really required when a string of symbols can be read in two or more ways *and* the different readings mean different things. So there is no need to use parentheses on the string of symbols 'C & E & I'; though we can group the conjuncts either as '(C & E) & I' or as 'C & (E & I)', there is no difference in meaning between these two formulae. However, we will later adopt some simple rules about how to use parentheses, and these rules will indeed require parentheses in 'C & E & I'. The reason for this is that following a universal rule mechanically is easier than qualifying it with a list of exceptions. It is also useful to settle on a particular occurrence of '&' in 'C & E & I' as the main connective.

The use of parentheses could be minimized by introducing a collection of conventions about how a formula is to be read if it lacks parentheses, in other words, by introducing a collection of conventions about the scope of connectives in otherwise ambiguous strings of symbols like (3*). For example, we could have a scope convention that '&' takes the smallest scope possible, which would force the reading 'C ∨ (E & I)' on (3*). But conventions for two-place connectives can be difficult to remember, so we will only have one scope convention, (SC~), for the one-place connective '~':

(SC~) '~' takes the smallest possible scope.

Hence although the string of symbols '~P & Q' can be read in two different ways, as '(~P) & Q' or as '~(P & Q)', (SC~) requires that we interpret it in the first of these two ways. If the second is what we want, then we must put the parentheses in explicitly.

In many cases, the exact symbolization of a sentence of English is not immediately obvious. For example, although

(5) Our budget deficit and our trade deficit will both rise

contains the connective 'and', the connective does not join two complete sentences, since the phrase 'Our budget deficit' on its left is not a complete sentence. But (5) means the same as a sentence in which 'and' does join two complete sentences, namely

(6) Our budget deficit will rise and our trade deficit will rise.

(5) is merely a more economical syntactic variant of (6); in English grammar, (5) is said to result from applying *conjunction reduction* to (6). In symbolizing sentences which contain sentential connectives that apparently do not join complete sentences, but which are simply economical variants of sentences where the connectives do join complete sentences, the first step is to recover the longer sentence in which the connectives join only complete sentences, such as (6); the second step is to choose sentence-letters for the atomic sentences which appear in the longer sentence; and then the last step is to write out the formula. Let us call the longer sentence in which the connectives join complete sentences the *full sentential form* of the original sentence. We have expanded (5) to its full sentential form (6), so if we use the sentence-letter 'B' for the atomic sentence 'our budget deficit will rise' and the sentence-letter 'T' for the atomic sentence 'our trade deficit will rise', we can symbolize (5) as

(5.s) B & T.

Notice, however, that there are examples rather like (5) in appearance but where expansion in the style of (6) would be wrong. For instance, 'Bill and Ted are friends' does not mean 'Bill is a friend and Ted is a friend'. The 'and' in this example is said to have a 'collective' sense, which is not expressed by '&'. In symbolizing a sentence as a conjunction, therefore, we must make sure that its 'and' really does join two complete sentences. 'Bill and Ted are friends' could only be symbolized by a single sentence-letter in sentential logic if justice is to be done to the 'and'.

Another common complication in symbolizing an English sentence is that it may contain a connective which is not one of the five connectives listed at the start of this chapter. However, the new connective may be expressible by one of our five or by some combination of them. The simplest example involves 'but'. Using 'J' for 'John will attend the meeting' and 'M' for 'Mary will attend the meeting', we can symbolize

(7) John will attend the meeting but Mary will not

as

(7.s) J & ~M

since for logical purposes there is no difference between 'and' and 'but': so far as who is at the meeting is concerned, (7) may as well say 'and' instead of 'but'.[1]

Negation is sometimes expressed not by 'not', but rather by some prefix attached to a word. For instance,

(8) Insider trading is unethical and illegal

contains two occurrences of 'not', one as 'un' and the other as 'il'. So using 'E' for 'insider trading is ethical' and 'L' for 'insider trading is legal', and expanding (8) to its full sentential form 'Insider trading is unethical and insider trading is illegal', we would symbolize it as

(8.s) $\sim E \ \& \sim L$.

Notice, however, that not every use of a prefix like 'un' can be treated as the sentential negation 'it is not the case that'. For example, 'an unintelligible book often sells well' does not mean the same as 'it is not the case that an intelligible book often sells well'. Each example must be considered on its own merits.

Just as negation need not be expressed by 'not', 'and' need not express conjunction. Consider the following:

(9) At least one of our two deficits, budget and trade, will rise.

Though (9) contains 'and', the crucial phrase is 'at least one', which has the effect of disjunction. Thus the full sentential form of (9), paraphrasing to make the occurrence of disjunction explicit, is simply

(10) Either our budget deficit will rise or our trade deficit will rise.

Using the same symbols as for (6), the symbolization would therefore be

(10.s) $B \vee T$.

However, the claim that (9) and (10) mean the same, that is, that (10) is a correct paraphrase of (9), may be challenged. Suppose that both our deficits rise. Then (9) is certainly true, but is (10)? Or does (10) carry the implication that one deficit or the other will rise, *but not both?* If it does, then (10) would be false if both deficits rise, and so cannot mean the same as (9). Sometimes the word 'or' seems to carry the implication 'but not both', for example in the prediction 'tonight I will either watch television or read a good book'; and sometimes it does not carry this implication, as in 'you can take an advanced logic course if either you have taken an introductory course or you have taken an advanced mathematics course'—obviously, one would not exclude a student

[1] Of course, there is *a* difference between 'and' and 'but'. (7) conventionally carries the suggestion that there is some antecedent expectation of Mary's accompanying John, whereas 'and' would not suggest this. But that difference does not affect what can be validly deduced from (7) (or what (7) can be validly deduced from), so it is not a difference of any *logical* import. It is not a difference in what would make the statement *true*, but only in what would make 'but' rather than 'and' *appropriate*.

from an advanced logic course on the grounds that she has taken *both* an introductory logic course and an advanced mathematics course! What are we to make of this apparently shifting nature of the meaning of ‘or’?

One explanation of the shift is that the English connective ‘either…or…’ is ambiguous between two senses, an *inclusive* sense, meaning at least one, and an *exclusive* sense, meaning one or the other but not both, that is, exactly one. The sense ‘either…or…’ has in any particular occurrence would then depend on context. In the exclusive sense of ‘or’, (10) would be false if both deficits rise, while in the inclusive sense, it would be true. However, there is another possible account of the disagreement. Perhaps those who hear (10) as false in the case where both deficits rise are confusing the question of whether or not (10) is true in this situation with whether or not it would be appropriate to assert (10) if one knew this was the situation. On this view, (10) would be *inappropriate* if both deficits were known to be rising—one should assert the conjunction instead—but it is still literally *true*.

If the first explanation of the disagreement over (10) is correct, then we should simply *decide* which of the two senses of ‘or’ we will use our symbol ‘∨’ to express. If the second explanation is correct, then the inclusive sense is the only sense of ‘or’. Since using ‘∨’ to express the inclusive sense is unobjectionable by both accounts, that is the use we will make. *In this book, then, ‘either…or…’ is always understood inclusively;* when we want to express the exclusive sense, we will explicitly use the words ‘and not both’.

Finally, there are English connectives which require a combination of our five to express. ‘Neither…nor…’ is a familiar example, as in

(11) Neither John nor Mary will attend the meeting.

To say that neither of the two will attend is to say that John will not attend *and* Mary will not attend (this unpacks the meaning of ‘neither…nor…’ and also gives a full sentential form). So using ‘J’ for ‘John will attend the meeting’ and ‘M’ for ‘Mary will attend the meeting’ as before, (11) comes out as

(11.s) ~J & ~M.

(11) should be distinguished carefully from

(12) John and Mary won’t both attend the meeting

which does not mean that they will *both* be absent, but rather, means that *at least one* will be absent (that is, at most one will attend). In other words, there is a significant difference between ‘won’t both attend’ and ‘both won’t attend’. (12) can be formalized equally naturally in two ways (still using ‘J’ and ‘M’ as above). If we follow the structure of the English, we have a formula in which ‘~’ has a conjunction within its scope, reflecting the occurrence of “won’t” in (12) preceding ‘both’. This gives

(12.s) ~(J & M).

On the other hand, if we use our paraphrase of (12) as 'at least one will be absent', where 'absent' is 'not in attendance', and expand the paraphrase to its full sentential form 'one will not attend or the other will not attend', the symbolization we get is

(12.s′) ~J ∨ ~M.

We will later explain what it is for two formulae to be *logically equivalent*, and (12.s) and (12.s′) will be an example of a pair of equivalent formulae. But (12.s) is our preferred symbolization of (12), since it follows (12)'s syntax more closely. It is by following the English syntax as closely as possible that we get a single preferred symbolization of an English sentence (*modulo* choice of sentence-letters). This procedure is also more likely to lead to a *correct* symbolization. Nevertheless, since (12.s) and (12.s′) are equivalent, there is no logical objection to using one rather than the other in symbolizing (12).

❑ Exercises

Symbolize the following sentences of English, saying what atomic sentences your letters stand for. Write out these sentences so that they are grammatically complete, and also explicit (replace words like 'they' and 'it' with the phrases which they stand for in the context). Be sure that your simple sentences do not contain any connective-words, and that you do not use two sentence-letters for what is essentially the same English sentence. Use the suggested sentence-letters where given, and use no connectives other than '~', '&' and '∨'.

(1) Van Gogh's pictures are the world's most valuable, yet they are not the most profound. (Use 'E', 'P')
(2) Van Gogh's pictures are not the world's most valuable, but they are the most profound.
(3) Van Gogh's pictures are neither the world's most valuable nor the most profound.
(4) Van Gogh's pictures aren't both the world's most valuable and the most profound.
(5) Neither digital computers nor neural networks can simulate every aspect of human intelligence, though each can simulate some. (D, N, E, O)
*(6) Even though inflation is falling, the government can't guide the economy wisely and at the same time regain its popularity.
(7) Smith, Brown and Robinson are politicians, yet all three of them are honest. (6 letters)
(8) Smith, Brown and Robinson are persuasive, though none of them is honest.
*(9) Smith, Brown and Robinson are lawyers—still, at least two of them are honest.
(10) Smith, Brown and Robinson are ethicists, but just one of them is honest.
(11) The favorite won't win, but not for want of trying. (W, T)
(12) It never rains but it pours. (R, P) [Hint: (12) is a saying to the effect that troubles don't come just in ones and twos but in large numbers.]

3 Conditionals and biconditionals

English has many ways of expressing the indicative conditional. So far we have met only the most straightforward 'if...then...' construction, as in

(1) If Smith bribes the instructor, then Smith will get an A

for which, using 'R' for 'Smith bribes the instructor' and 'G' for 'Smith will get an A', we have the formula

(1.s) $R \rightarrow G$.

In a conditional formula such as (1.s), the part or *subformula* on the left of the arrow is the antecedent and the subformula on the right, the consequent. So the antecedent is the first subformula we meet. But in English, the antecedent does not have to come first. (1) means exactly the same as

(2) Smith will get an A if Smith bribes the instructor

and has exactly the same symbolization (1.s). In (1) and (2), the antecedent is the subsentence which follows the 'if', which in an inversion like (2) is not the subsentence which comes first.

Another way of expressing the same idea as (1) and (2) is

(3) Smith's bribing the instructor is a *sufficient condition* for Smith to get an A.

One condition is sufficient for another when the first's obtaining is *all* that is required for the second's obtaining, and since (1) and (2) both say that all that is required for getting an A is bribing the instructor, (1) and (2) mean the same as (3). But we do not symbolize (3) using sentence-letters for 'Smith's bribing the instructor' and 'Smith to get an A', since these phrases are not complete sentences. The complete sentence which corresponds to the condition of Smith bribing the instructor is just 'Smith bribes the instructor', and similarly, the sentence which corresponds to the condition of Smith getting an A is 'Smith gets an A'. So we can use 'R' and 'G' again and (3) is symbolized by (1.s), just like (1) and (2).

(1), (2) and (3) contrast with

(4) Smith will get an A *only* if Smith bribes the instructor

and

(5) Smith's bribing the instructor is a *necessary* condition for Smith to get an A.

One condition is said to be *necessary* for another when the obtaining of the first is *required* for the obtaining of the second. Thus (5) says that a bribe is required for getting an A, which is what (4) says as well. This means that there are two ways in which (4) and (5) differ in meaning from (1), (2) and (3). First, whereas (1), (2) and (3) say that *all* Smith has to do to get an A is to bribe the instructor, (4) and (5) do not say this: they both leave open the possibility that there is *more* Smith must do to get an A besides give the bribe. Hence (1), (2) and (3) would all be false in a situation where Smith's instructor demands both a bribe and regular class attendance before he gives anyone an A, while (4) and (5) would be true in such a situation.

On the other hand, there is something (1), (2) and (3) leave open that (4) and (5) do not, and that is whether there is a way of getting an A *without* bribing the instructor, say by doing excellent work. (4) and (5) exclude this possibility, but (1), (2) and (3) do not. So how should (4) and (5) be symbolized? There is a natural way of paraphrasing (4) and (5) which answers this question, for both say the same as

(6) If Smith does not bribe the instructor he will not get an A,

that is,

(6.s) $\sim R \rightarrow \sim G$.

Another approach is suggested by the thought that (4) seems to say that if Smith gets an A then he has bribed the instructor. Though 'bribes' changes to 'has bribed' in this paraphrase, we can abstract from this. Thus as an alternative to (6.s) we could also have

(6.s′) $G \rightarrow R$.

(6.s′) is sometimes said to be the *converse* of (1.s), and (6.s) is said to be the *contrapositive* of (6.s′), though we will not make much use of these terms. Since (6.s) and (6.s′) are correct symbolizations of English sentences with the same meaning, those formulae have the same meaning as well, and since (1.s) symbolizes a sentence with a different meaning, it should turn out to mean something different from (6.s) and (6.s′).

For the sake of having a definite policy, we should decide between (6.s′) and (6.s) for dealing with 'only if'. In some ways (6.s) is a more natural means of expressing the effect of 'only if', but when we come to manipulate formulae it will be an advantage to work with the simplest ones possible. Since (6.s′) is simpler than (6.s), we will use it as the pattern for 'only if', as stated in the following rule. Let p and q be any two sentences of English. Then

- 'if p then q' and 'q if p' are both symbolized '$p \rightarrow q$'
- 'q only if p' is symbolized '$q \rightarrow p$'

The rule for 'only if' must be carefully memorized. Probably the most common error in symbolizing sentences of English in sentential logic is to symbolize 'only if' sentences the wrong way around. Notice, by the way, how the letters '*p*' and '*q*' are used in stating this rule. We are using them to make a generalization which is correct for all English sentences, in the same way that mathematicians use the letters '*x*' and '*y*' to make generalizations about numbers, as in 'for any numbers x and y, if x and y are even so is $x + y$'. '*x*' and '*y*' are sometimes called *numerical variables*, because they stand for numbers, but not any particular numbers fixed in advance; so they contrast with numerals like '0' and '5', which do stand for particular numbers. Similarly, '*p*' and '*q*' are *sentential variables*, because they do not stand for fixed sentences, by contrast with sentence-letters like 'A' and 'B'.

There are a number of other locutions used in English to express indicative conditionals, such as 'provided' and 'whenever', and to determine exactly what conditional is being expressed it is often useful to ask what is being said to be a *necessary condition* for what or what is being said to be a *sufficient condition* for what. Thus

(7) Smith gets an A whenever he bribes the instructor

says that Smith's bribing the instructor is a sufficient condition for getting an A, since according to (7), that is all he has to do to get an A. Consequently, (7) amounts to the same as (1), and so has the symbolization (1.s).

(8) Smith will get an A provided he bribes the instructor

also says that a bribe is sufficient for an A and again is symbolized as (1.s). In general, when a new way of expressing a conditional construction is encountered, we ask what is being said to be sufficient for what, or necessary for what, and then give the appropriate translation into symbols using the symbolizations of (3) and (5) as a guide.

The difference between necessary and sufficient conditions is encountered often enough in real life. For example,

(9) Your car will start only if there is fuel in its tank

says, correctly, that having fuel in its tank is necessary for your car to start. If we use 'S' for the atomic sentence 'your car will start' and 'F' for the atomic sentence 'your car has fuel in its tank' we can symbolize (9) following the pattern of (4) and (5) as

(9.s) $S \rightarrow F$.

Again, it helps to see why this is right if we note that (9.s) could be read back into English as 'if your car starts then (it follows that) it has fuel in its tank'. For its having fuel in its tank to be a consequence of its starting, its having fuel in its tank must be necessary for it to start.

While (9) is in most circumstances true, and will be until electric cars become more common,

(10) Your car will start if it has fuel in its tank

is very often false. It is not *sufficient* for a car to start that it have fuel in its tank—usually when one's car refuses to start, it does have fuel in its tank (however, a mechanic who has checked that everything *else* is working would be in a position to assert (10)). (10)'s symbolization, on the model of (2), is

(10.s) $\mathrm{F} \rightarrow \mathrm{S}$.

One other common English connective which can be captured by the conditional is 'unless', as in

(11) Smith will fail unless he bribes the instructor

and

(12) Your car will not start unless it has fuel in its tank.

It is not too difficult to see that (11) and (12) can be accurately paraphrased respectively by the following:

(11.a) If Smith does not bribe the instructor then he will fail

and

(12.a) If your car does not have fuel in its tank, it will not start.

In these paraphrases, the *negation* of the sentence immediately following the 'unless' becomes the antecedent of the conditional, while the other constituent subsentence becomes the consequent (in an 'unless' sentence, this subsentence does not have to precede the 'unless', in view of the grammatical acceptability of, for instance, 'unless he bribes the instructor, Smith will fail'). So using 'L' for 'Smith fails', the symbolizations of (11) and (12) at which we arrive are:

(11.s) $\sim\mathrm{R} \rightarrow \mathrm{L}$

and

(12.s) $\sim\mathrm{F} \rightarrow \sim\mathrm{S}$.

It is important to note that (11) does not say that bribing the instructor is sufficient for not failing, so '$\mathrm{R} \rightarrow \sim\mathrm{L}$' would be wrong. Similarly, (12) does not say that having fuel in its tank is sufficient for your car to start, so '$\mathrm{F} \rightarrow \mathrm{S}$' would be wrong. The rule we extract from this discussion is therefore:

- for any sentences p and q, 'p unless q' and 'unless q, p' are symbolized '$\sim q \rightarrow p$'

The last of our five basic connectives is the biconditional, '…if and only if…', for which the symbol is '↔'. Suppose Smith's instructor will only give an A to a student who bribes him but requires nothing more of such a student. Then (2) and (4) are both true, and we could express this simply by writing their conjunction

(13) Smith will get an A if he bribes the instructor and Smith will get an A only if he bribes the instructor

for which we have the symbolization

(13.s) $(R \rightarrow G) \& (G \rightarrow R)$.

But (13) is rather long-winded, and by using '…if and only if…' we can avoid its repetitious aspect:

(14) Smith will get an A if and only if he bribes the instructor.

(13) and (14) both say that a bribe is necessary *and* sufficient for getting an A; the 'if' captures sufficiency while the 'only if' captures necessity. (14) could be symbolized by (13.s), but instead we use the special symbol '↔', which allows us to contract (13.s) into

(14.s) $G \leftrightarrow R$.

We think of the symbol '↔' in a slightly different way from the others. The others were introduced as symbols for connectives having meaning in their own right, but we treat '…if and only if…' as merely as an abbreviation for '…if…and …only if…'. In other words, (14) abbreviates (13) and so (14.s) is an abbreviation of (13.s). This different attitude toward '↔' will have certain consequences in later chapters.

Like the conditional, the biconditional can be expressed in English in more than one way. A common variant of '…if and only if…', at least in mathematical writing, is '…just in case…', as in

(15) Smith will get an A just in case he bribes the instructor

which also receives the symbolization (14.s). Other variants of '…if and only if…' will be encountered in the exercises.

Our collection of connectives allows for the symbolization of sentences of some complexity, for example

(16) John will study hard and also bribe the instructor, and if he does both then he'll get an A, provided the instructor likes him.

In approaching the symbolization of a complicated compound sentence, the first step is to classify the whole sentence as a conjunction or conditional or whatever, and then to identify the atomic sentences which occur in it. Reading through (16) we see that it is a conjunction whose first conjunct is another conjunction, 'John will study hard and also bribe the instructor', and whose second conjunct is some kind of complicated conditional construction. As for the atomic sentences which occur in it, we have

S: John will study hard
R: John will bribe the instructor
G: John will get an A
L: The instructor likes John

Here we list the relevant atomic sentences alongside the sentence-letter we have chosen for each. Such a list is called a *dictionary*, and every solution to a symbolization problem should begin with one. Note also that the English phrases are *complete* sentences: we do not write 'R: bribe the instructor', since 'bribe the instructor' is not a complete sentence. In addition, no words whose meaning or reference is settled by linguistic context appear in the dictionary: we do not write 'L: The instructor likes him' since the reference of 'him' is clear from context: looking at (16), we see that 'him' refers to John, so we make this explicit in the dictionary.

Once the dictionary has been given, the best way to proceed is to go through the sentence to be translated and substitute letters for atomic sentences, leaving the connectives in English. This intermediate step makes it more likely that the final formalization we arrive at will be correct. So using the dictionary to substitute in (16), we obtain

(16.a) S and R, and if S and R then G, provided L.

Note that we handle 'he does both' by expanding it to full sentential form: 'he does both' is simply an economical way of repeating 'he studies hard and bribes the instructor'.

The remaining problem is to place 'provided L' properly in the formula. The effect of 'provided L' is to say that the joint truth of S and R is sufficient for the truth of G *if* L is also the case. In other words, if L is the case, the truth of S and R suffices for the truth of G. So the most natural formalization is produced by rewriting (16.a) as

(16.b) S and R, and if L, then if S and R then G.

(16.b) is a conjunction. Its first conjunct is 'S and R', its second, 'if L, then if S and R then G'. Symbolizing 'S and R' is trivial. 'If L, then if S and R then G' is a conditional with antecedent 'L' and consequent 'if S and R then G'. So as another intermediate step on the way to a final answer, we can rewrite (16.b) as

(16.c) (S & R) & (L → (if S and R then G)).

'If S and R then G' is symbolized '(S & R) → G', so substituting this in (16.c) we get the following full symbolization of (16),

(16.s) $(S \,\&\, R) \,\&\, (L \rightarrow ((S \,\&\, R) \rightarrow G))$.

This procedure illustrates the best way of approaching complicated symbolization problems, which is to use a step-by-step method. If one attempts to write down the formula straight off, without any intermediate steps, it is very likely that something will go wrong. As a final check, we should inspect for unmatched parentheses: we count the number of left parentheses and make sure there is an equal number of right parentheses. The step-by-step procedure by which we arrived at (16.s) guarantees that the parentheses are in the correct positions.

❑ Exercises

Symbolize the following sentences of English, giving a dictionary for each example. Write down the simple English sentences so that they are grammatically complete and also explicit (apart from (1) and (2), replace words like 'they' and 'it' with the phrases which they stand for in the context of the compound sentence). Be sure that your simple sentences do not contain any connective-words, and that you do not use two sentence-letters for what is essentially the same English sentence.

(1) If it's Tuesday then it must be Belgium. (T, B)
(2) If it's Tuesday and it's not Belgium, I'm lost. (T, B, L)
(3) If the military relinquishes power, then the new government will release all political prisoners if it has any respect for human rights.
(4) If the economy declines, then unless there is a change of leadership there will be a recession. (D, C, R)
*(5) Grades may be posted provided students so request and are not identified by name. (P, R, S)
(6) The applicants may examine their dossiers only if they have not already waived their right to do so and their referees approve.
(7) If enough people show up we will take a vote, otherwise we will reschedule the meeting.
*(8) One has the right to run for president if and only if one is an American citizen.
(9) The unrestricted comprehension principle is true just in case arithmetic is inconsistent.
(10) Not only will John and Mary have to live on campus if they aren't local residents, but neither will be allowed to park on campus—whether or not they are local residents.
(11) The award of the license is conditional upon passing the exam. (A: The license is awarded)
*(12) All that's required to pass the exam is that one makes an effort. (E: One makes an effort)
(13) The exam can't be passed without making an effort. (P: One passes the exam)

(14) If, but only if, they have made no commitment to the contrary, may reporters reveal their sources, but they always make such a commitment and they ought to respect it.

*(15) An increase in government funding is a necessary condition of an improvement in the quality of education, but not a sufficient condition. (F: There is an increase in government funding)

(16) A right to smoke in public is contingent upon such activity's not significantly affecting the health of others, but nowadays we know that the idea that it doesn't is wrong.

4 Symbolizing entire arguments

So far we have symbolized only single sentences, but the problems which arise in symbolizing entire arguments are similar. In this section we discuss four examples.

> A: If Smith bribes the instructor then he'll get an A. And if he gets an A potential employers will be impressed and will make him offers. But Smith will receive no offers. Therefore Smith will not bribe the instructor.

The first step in symbolizing an argument is to identify its conclusion, which is often, but not always, the last complete sentence in the argument. Conclusions are usually indicated by the presence of words or phrases like 'therefore', 'so it follows that', 'hence', or as in Example C on the next page, 'so'. All these *conclusion indicators* may be abbreviated by '∴'. The conclusion in A is 'Smith will not bribe the instructor'.

Next, we need to identify the atomic sentences in the argument and construct a dictionary, *taking care to avoid duplication*. That is, we should not introduce a new sentence-letter for what is essentially a recurrence of the *same* atomic sentence. Reading through A closely indicates that the following atomic sentences occur in it, which we list in dictionary format:

R: Smith bribes the instructor
G: Smith gets an A
E: Potential employers will be impressed
O: Potential employers will make Smith offers

We then go through the argument symbolizing each premise and the conclusion in turn. The first premise is (3.1) ((1) from §3), and is symbolized as (3.1.s), '$R \rightarrow G$'. The second premise with letters substituted for its constituent atomic sentences is

(1) If G, E and O

and so the formula is

(1.s) G → (E & O).

We do not symbolize the initial 'And' of the second premise, since it is present in the English only for stylistic reasons, to aid the flow of the passage. The third premise does not get its own sentence-letter, 'C', for 'Smith will receive offers', since in the context of this argument, 'Smith will receive offers' is simply another way of saying 'potential employers will make Smith offers'. Hence the third premise says that these employers will not make him offers, and so is symbolized as '~O'. The conclusion is '~R', and we therefore have the whole argument symbolized as

B: R → G
G → (E & O)
~O
∴ ~R

Our second example is a little more complex:

C: If logic is difficult, then few students will take logic courses unless such courses are obligatory. If logic is not difficult then logic courses will not be obligatory. So if a student can choose whether or not to take logic, then either logic is not difficult or few students take logic courses.

The conclusion in C is the complex conditional 'If a student can choose whether or not to take logic, then either logic is not difficult or few students choose to take logic courses'. The following atomic sentences occur in C, which we list in dictionary format:

T: Logic is difficult
F: Few students take logic courses
O: Logic courses are obligatory

We do not use a separate sentence-letter for 'such courses are obligatory' since 'such courses' simply means 'logic courses'; and we do not use a separate sentence-letter for 'a student can choose whether or not to take logic' since the meaning of this sentence does not differ significantly from that of 'Logic courses are not obligatory', for which we have '~O'.

Turning to the symbolization of the premises and conclusion, we see that the first premise contains 'unless'. Applying our rule on page 23 for this connective, where *p* is 'F' and *q* is 'O', we obtain

(2) If T then (~O → F)

and hence for the whole premise,

(2.s) T → (~O → F).

The second premise is ‘if not-T then not-O’ and the conclusion is ‘if not-O then either not-T or F’. So we can give the final symbolization of the whole argument as follows:

D: $T \rightarrow (\sim O \rightarrow F)$
$\sim T \rightarrow \sim O$
$\therefore \sim O \rightarrow (\sim T \vee F)$

Now an example with still more complexity:

E: If God exists, there will be no evil in the world unless God is unjust, or not omnipotent, or not omniscient. But if God exists then He is none of these, and there is evil in the world. So we have to conclude that God does not exist.

As before, we begin by identifying the conclusion of E, which is indicated by the phrase ‘so we have to conclude that’; ‘so we have to conclude that’ is simply a wordier version of ‘so’, and thus E’s conclusion is the negative sentence ‘God does not exist’.

Next, we identify the atomic sentences in the argument and construct a dictionary, again taking care to avoid duplication: we do not want to introduce a new sentence-letter for a statement whose meaning can be expressed, perhaps using connectives, by sentence-letters already in the dictionary. Reading through E closely suggests the following dictionary:

X: God exists
V: There is evil in the world
J: God is just
M: God is omnipotent
S: God is omniscient

Are there any other atomic sentences in E? ‘God is none of these’ is presumably not atomic because ‘none’ contains the connective ‘not’; ‘God is none of these’ means ‘it is not the case that God is some of these’. So should ‘God is some of these’ be in our dictionary? Again the answer is no, for ‘God is some of these’, like ‘he does both’ in (3.16), is simply a way of getting the effect of repeating sentences without actually repeating their very words. In this case, ‘God is some of these’ is an economical version of ‘God is unjust, or God is not omnipotent or God is not omniscient’, which is a disjunction of negations of atomic sentences already listed in the dictionary (we will group the last two disjuncts together). It would therefore be a mistake to introduce a new sentence-letter for ‘God is some of these’. Note also that we use ‘or’ rather than ‘and’ in the full sentential form of ‘God is some of these’. Using ‘and’ would result instead in the full sentential form of ‘God is *all* of these’.

We can now symbolize the argument premise by premise. We start with the first premise and substitute the sentence-letters of our dictionary into it. This gives us

(3) If X then (~V unless (~J or (~M or ~S)))

and symbolizing the 'if…then…' yields

(3.a) X → (~V unless (~J or (~M or ~S))).

Next, we recall our rule for 'unless', that '*p* unless *q*' becomes 'if not-*q* then *p*'; here the '*p* unless *q*' component is '~V unless (~J or (~M or ~S))', and so *p* is '~V' and *q* is '~J or (~M or ~S)' (note again the similarity between our use of the sentential variables '*p*' and '*q*' to stand for any sentences and the mathematician's use of '*x*' and '*y*' to stand for any numbers). Therefore we can rewrite (3.a) as the intermediate form

(3.b) X → (if not-[~J or (~M or ~S)] then ~V).

Putting the remaining connectives into symbols produces

(3.s) X → (~[~J ∨ (~M ∨ ~S)] → ~V).

(3.b) nicely illustrates the use of brackets rather than parentheses as an aid to readability, since it makes it easier to see which left marker belongs with which right one.

The second premise of E is 'if God exists then he is none of these, and there is evil in the world', which is a conjunction with 'there is evil in the world' as second conjunct. To say God is none of these, in the context of our argument, is to say that he is not unjust, not not-omnipotent and not not-omniscient. Although the repeated 'not's cancel each other, in what is known as a *double-negative*, we will retain them in our symbolization for the sake of being faithful to the English, which also contains double-negatives (in the combination of 'none' with '*un*just', 'not omnipotent', etc.). So when the second premise is expanded to full sentential form and substituted into from the dictionary, we obtain

(4) If X then not not-J and not not-M and not not-S, and V

which fully symbolized becomes

(4.s) [X → (~~J & (~~M & ~~S))] & V.

Note the role of the brackets in ensuring that (4.s) is a conjunction; without them it would be possible to read the formula as a conditional with V as the second conjunct of its consequent.

We can now write out the whole argument in symbols:

F: X → [~(~J ∨ (~M ∨ ~S)) → ~V]
[X → (~~J & (~~M & ~~S))] & V
∴ ~X

F gives the sentential logical form of E, and with the techniques to be developed in the next chapter we will be able to determine whether or not this argument-form is valid, and hence whether or not E is sententially valid.

Our last example is quite tricky:

> G: We can be sure that Jackson will agree to the proposal. For otherwise the coalition will break down, and it is precisely in these circumstances that there would be an election; but the latter can certainly be ruled out.

G illustrates a common phenomenon in ordinary argumentative discourse, which is that the conclusion need not always be the *last* proposition to be stated. A careful reading of G indicates that it is 'Jackson will agree to the proposal' and *not* 'the latter can be ruled out' which is the conclusion of the argument: what follows the opening sentence of G is a list of the *reasons why* Jackson will agree, in other words, what follows is a list of the premises from which 'Jackson will agree' is supposed to follow. Phrases like 'for' and 'after all' are often used to introduce the premises for a conclusion which has already been stated. Another clue that the conclusion of G occurs at its start is the use of 'we can be sure that', which is a conclusion indicator like 'we have to conclude that' in E.

What atomic sentences occur in G? We have already identified 'Jackson will agree to the proposal'. Evidently, 'the coalition will break down' and 'there will be an election' also occur. 'The latter can be ruled out' is not a new atomic sentence, since 'the latter' refers to 'there will be an election', so that 'The latter can be ruled out' is just another way of saying that there will not be an election. Hence we need the following dictionary:

J: Jackson will agree to the proposal
C: The coalition will break down
E: There will be an election.

The first conjunct of the first premise is 'otherwise the coalition will break down', where the 'otherwise' refers to what has already been stated, that Jackson will agree. To say that *otherwise* the coalition will break down is to say that if he does *not* agree, the coalition will break down. Substituting into the first conjunct from the dictionary and symbolizing the connectives then gives

(5.s) $\sim J \rightarrow C$

as the formula.

The second conjunct of the first premise is 'it is precisely in these circumstances that there would be an election'. Rereading G we see that by 'these circumstances' is meant the circumstances of the coalition breaking down, so that the second conjunct, in full sentential form, says that it is precisely if the coalition breaks down that there will be an election. What is the import of 'precisely if'? Since the best way to assess the meaning of an unfamiliar connective which embodies some kind of conditional construction is to express it in terms of necessary and sufficient conditions, we ask what it is that the second premise says is necessary, or sufficient, for what. According to the second conjunct of premise 1, there *would* be an election in the circumstances of the coalition breaking down, so a breakdown is being said to be *sufficient* for the occurrence of an election. Part of the meaning of the second conjunct is therefore 'if the coalition breaks down there will be an election'. But that is not the whole meaning, since the import of 'precisely' is that there are no *other* circumstances in which an election would occur; hence, a breakdown of the coalition is necessary for an election as well. Consequently, we need the biconditional to get the full effect of 'precisely if'. The second conjunct of the first premise is:

(6) There will be an election if and only if the coalition breaks down

or in symbols, '$E \leftrightarrow C$'. As we already remarked, the conclusion of G is 'Jackson will agree to the proposal', so the logical form at which we arrive for the whole argument is

H: $(\sim J \rightarrow C) \& (E \leftrightarrow C)$
$\sim E$
$\therefore J$

Two questions about this argument are worth pursuing. First, would it have made any difference if instead of formalizing the whole argument with two premises, the first of which is a conjunction, we had formalized it with three premises, breaking the first premise up into its conjuncts and treating each as a separate premise? This would have given us

I: $(\sim J \rightarrow C)$
$(E \leftrightarrow C)$
$\sim E$
$\therefore J$

Intuitively, the answer should be that as far as validity is concerned, there is no difference between H and I, since a conjunction contains exactly the same information as its two conjuncts. This is just to say that a conjunction is true if and only if its two conjuncts are true. However, as we shall see in Chapter 3, the equivalence between H and I is a special feature of conjunction.

Secondly, though (6) seems the natural way of expressing the second conjunct of premise 1, would it have made any difference if we had given the full sentential form as

(7) The coalition will break down if and only if there is an election

instead? According to (6), the coalition's breaking down is necessary and sufficient for there being an election, and what makes (6) natural is that it expresses the relationship in the direction of causality: it is a breakdown which would cause an election and not vice versa. However, 'if and only if' is not tied to expressing causal relationships: it simply abbreviates the conjunction of two conditionals, and the same two are conjoined in (6) and (7), though in different orders. But since the order of conjuncts does not affect what a conjunction says, (6) says the same as (7).

Another way of seeing this is to note that if a breakdown is sufficient for an election, as (6) says, then an election is necessary for a breakdown, as (7) says, in the sense that a breakdown cannot occur unless an election does too (because a breakdown would cause an election). Equally, if a breakdown is necessary for an election, as (6) says, an election is sufficient for a breakdown, as (7) says, in the sense that if an election occurs, that is all the information we need to conclude that a breakdown has also occurred. So from (6) it follows that an election's occurring is sufficient and necessary for a breakdown to have occurred, which is exactly what (7) says. Consequently we could just as well have '$C \leftrightarrow E$' in place of '$E \leftrightarrow C$'.

These last two examples illustrate our earlier warning that there is a certain license in speaking of 'the' sentential form of an argument. Keeping as close as possible to English structure will very often produce a 'preferred' form, but sometimes there will still be degrees of freedom, as, for example, in G, over whether to use '$C \leftrightarrow E$' or '$E \leftrightarrow C$', and in F, over how to parenthesize its three-disjunct disjunctions and three-conjunct conjunctions. This is why our official definition of 'sententially valid English argument' requires only that the argument have *a* sententially valid form. Fortunately, there is no harm in continuing to avail ourselves of the convenient fiction that there is always a unique preferred form for an argument, since it is never the case that of two equally acceptable symbolizations of an argument in sentential logic, one is valid and the other invalid.

❑ Exercises

Symbolize the following arguments, giving a complete dictionary for each. Be sure not to use different sentence-letters for what is essentially the same simple sentence.

(1) If the government rigs the election there will be riots. However, the government will play fair only if it is guaranteed victory, and isn't guaranteed victory unless it rigs the election. So there will be riots. (G, R, V)

(2) If the price of oil rises then there will be a recession, and if there is a recession, interest rates will fall. But interest rates will fall only if the government allows them to, and the government will keep them high if the price of oil rises. So either the price of oil won't rise or interest rates will fall.

(3) The players will go back to work if agreement is reached about their salaries, but this will be achieved, if at all, only if some of them take early retirement. So the players will not go back to work unless some retire early.

*(4) If Homer did not exist, it follows that the *Odyssey* was written by a committee or, if Butler was right, that it was written by a woman. But it was not written by a woman. So it was written by a committee.

(5) If I know I exist then I exist. I know I exist if I know I think, and I know I think if I think. But I do think. Therefore I exist.

(6) If God knows today what I will do tomorrow, then what I do tomorrow is foreordained. And if it's foreordained, then either I have no freedom of choice, or else I will freely choose to do what's foreordained. However, the latter is impossible. Therefore I have no freedom of choice unless God doesn't know today what I will do tomorrow.

(7) If internment is ended, the IRA will negotiate. But they will negotiate if either they lose Catholic support or the UDA disbands, and not otherwise. And the UDA will disband only on condition that internment is not ended. So if it isn't ended, then provided the IRA negotiate, they will keep Catholic support.

(8) John will not get a seat at the front of the class unless he arrives on time. The instructor will call roll right at the beginning of class provided she herself is not late. If she's not on time and John does get a seat at the front, he'll be able to finish his homework. So, if John isn't late in getting to class, then if the instructor does not call role right at the beginning of class, then he will get his homework done.

*(9) If parking is prohibited in the center of town then people will buy bicycles, in which case the general inconvenience is offset by the fact that the city shops will pick up some business. However, shops go bankrupt in town centers with restrictive parking policies. So it looks as though bicycles don't attract people. (P, B, O, U)

(10) Suppose no two contestants enter; then there will be no contest. No contest means no winner. Suppose all contestants perform equally well. Still no winner. There won't be a winner unless there's a loser. And conversely. Therefore, there will be a loser only if at least two contestants enter and not all contestants perform equally well.

(11) If there is intelligent life on Mars, then provided it takes a form we can recognize, we would have discovered it by now. If we are unable to recognize intelligent Martians, that must be because they are very different from ourselves. Since we have not, in fact, discovered any intelligent form of life on Mars, it appears, therefore, that either there are no intelligent Martians, or there are some, but they are very unlike us.

(12) The one thing that would defeat the Mayor is a financial scandal involving members of the council. After all, his survival is conditional upon the support of the urban middle class, which in turn depends upon the council's reputation for financial integrity. (D, S, U)

5 The syntax of the formal language

Before leaving the topic of symbolizing English we should be more exact about such notions as the *scope* of a connective in a formula and of the *main connective* of a formula, concepts we have already used without being completely precise about their application. Precise definitions of these concepts are possible only when we are equally precise about the *grammar* of the language into which we are translating English. Of course, it is easy to recognize that such a sequence of symbols as '~& B C' is ungrammatical, since there is no grammatical English into which it could be translated, even given a dictionary for the sentence-letters. But that is not an *intrinsic* account of the problem with '~& B C'. In general, one does not classify the sentences of a language other than English as grammatical or ungrammatical according to whether or not they can be translated into grammatical English. Other languages possess their *own* grammars, and it is with respect to these that judgements of grammaticality and ungrammaticality should be made. We are going to regard the *formal* language of our symbolizations as an independent language with its own grammar, just like other natural languages (Sanskrit, Spanish etc.). We will call this formal language the *Language of Sentential Logic*, or LSL for short.

There is one respect in which LSL is unlike a natural language. In natural language, the basic elements, words, have their own meanings, but the sentence-letters of LSL do not have their own meanings. A natural language is said to be an *interpreted* language, while a formal language is said to be *uninterpreted* (apart from its connectives). We can regard the dictionaries of our translations as temporarily bestowing a meaning on particular sentence-letters, so that it makes sense, within the context of a given problem, to ask if a complex sentence of LSL and a sentence of English mean the same. But the purpose of translation from a natural language into LSL is not the usual purpose of translation between two natural languages, which is to find a way of expressing the same meaning in both. The main purpose of symbolization, rather, is to exhibit the logical form of the English by rendering it in a notation that abstracts away from aspects of the English that may obscure its logical properties.

The collection of basic elements of a language is called the language's *lexicon*. The lexicon of English is the set of all English words, but the lexicon of LSL will differ from a natural language's lexicon in two respects. First, the basic elements of formulae of LSL are sentence-letters, not words. Second, in any natural language the number of words is finite, but we will allow the lexicon of LSL to contain infinitely many sentence-letters.

The lexicon of LSL:

> All sentence-letters 'A', 'B', 'C',..., 'A′', 'B′', 'C′',..., 'A″',...; connectives '~', '&', '∨', '→', '↔'; and punctuation marks '(' and ')'.

Next, we have to specify the grammatical rules, or syntax, of LSL. Formulae of LSL are constructed from sentence-letters and connectives with the aid of punctuation marks. A formula such as '(~A & ~B)' can be regarded as having been built up by stages. At the first stage, we choose the sentence-letters we are going to use, in this case 'A' and 'B'. At the next stage, we prefix each sentence-letter with the negation symbol. This gives us two grammatical formulae, '~A' and '~B', and we call the operation we apply at this stage *negation-formation*. Then we combine the two formulae into a single formula using '&' and parentheses to obtain '(~A & ~B)'; we call the operation used here *conjunction-formation* (the other syntactic formation operations we have at our disposal are disjunction-formation, conditional-formation and biconditional-formation). Each syntactic operation applies to grammatical formulae and produces a grammatical formula. Since the formulae which a construction process begins with are always grammatical formulae, sentence-letters being grammatical formulae by definition, and since at every subsequent step we apply some formation operation which always produces grammatical formulae if applied to grammatical formulae, it follows that at the end of a construction process we will always have a grammatical formula. So we can define the grammatical formulae as all and only the formulae produced by such a process.

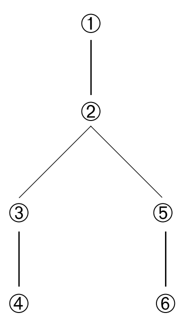
Some terminology and abbreviations will be useful. Instead of 'grammatical formula' it is usual to employ the phrase 'well-formed formula', abbreviated 'wff', plural 'wffs'. The formation operations are commonly called *formation rules*, and we abbreviate the five as (*f*-~), (*f*-&), (*f*-∨), (*f*-→) and (*f*-↔). Each rule is a conditional: *if* it is given a wff or wffs as input, then such and such a wff results as output. As remarked, this means that since the construction process always begins with wffs, it always ends with a wff. A *symbol-string* is a sequence of symbols from the lexicon of LSL, including ungrammatical ones like '~&ABC'. We may now define the collection of wffs by the following stipulations:

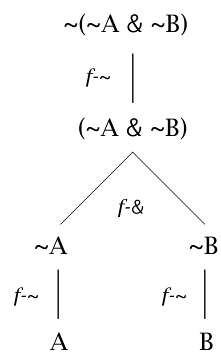
The formation rules of LSL:

(*f-sl*): Every sentence-letter in the lexicon of LSL is a wff.
(*f*-~): For any symbol-string *p*, if *p* is a wff then the result of prefixing '~' to *p* is a wff.
(*f*-&): For any symbol-strings *p* and *q*, if *p* and *q* are wffs, the result of writing '(' followed by *p* followed by '&' followed by *q* followed by ')' is a wff.
(*f*-∨): For any symbol-strings *p* and *q*, if *p* and *q* are wffs, the result of writing '(' followed by *p* followed by '∨' followed by *q* followed by ')' is a wff.
(*f*-→): For any symbol-strings *p* and *q*, if *p* and *q* are wffs, the result of writing '(' followed by *p* followed by '→' followed by *q* followed by ')' is a wff.
(*f*-↔): For any symbol-strings *p* and *q*, if *p* and *q* are wffs, the result of writing '(' followed by *p* followed by '↔' followed by *q* followed by ')' is a wff.
(*f!*): Nothing is a wff unless it is certified as such by the previous rules.

In writing these rules we are again using the letters '*p*' and '*q*' as sentential variables so that we can state generalizations about LSL formulae, analogous to generalizations about numbers such as ‘if *x* and *y* are even then *x* + *y* is even’. Clause (*f!*) is called the *closure condition:* it ensures that *only* the symbol strings derivable by the rules count as wffs. The other rules themselves ensure that *all* symbol strings derivable by them are wffs. To say that a symbol string is ‘certified’ as a wff by the formation rules is just to say that the string can be constructed by a sequence of applications of the rules. Note that in applying the rules for the binary connectives, each new formula is formed with outer parentheses around it, while (*f*-~) does not require outer parentheses. This reflects the fact that we have the scope convention for ‘~’ stated on page 15, so that outer parentheses are not needed to delimit a negation symbol’s scope.

It is possible to draw a picture of the construction of a wff. Such a picture is given in the form of an (inverted) tree, called a ‘parse tree’ by linguists. In general, a tree is a collection of nodes joined by lines in a way that determines various paths through the tree from a unique *root* node to various end nodes, called *leaves*. In an *inverted* tree, the root node is at the top and the leaves are at the bottom, as in the tree displayed below on the left. The root node of this tree is ① and its leaves are nodes ④ and ⑥. A *path* is a route from one end of the tree to the other in which no node is traversed twice, so the paths in our example are ①-②-③-④ and ①-②-⑤-⑥. A *branch* in a tree is a segment of a path *h* beginning with a node on *h* where paths diverge and including all subsequent nodes on *h* to *h*’s leaf. In the example on the left below, the branches are ②-③-④ and ②-⑤-⑥.

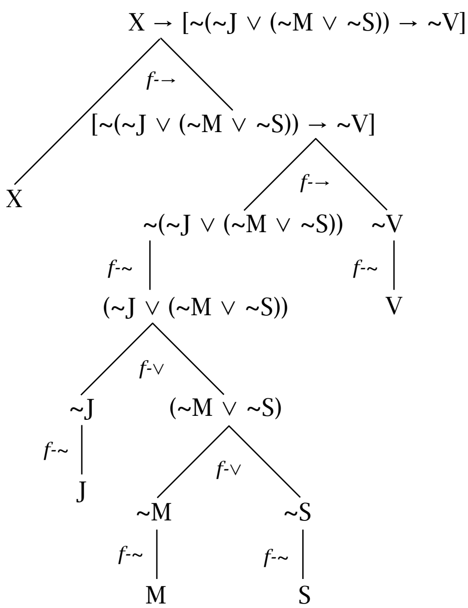




Parse trees are a special kind of tree in which the nodes are labeled with formulae in such a way that reading the tree from bottom to top recapitulates the construction of the formula. The sentence-letters label the leaves of the tree and the final formula labels the root. We also indicate which formation rule has been used in the passage up the tree to a particular node. On the right above we display a parse tree that results from labeling the root node of the tree on its left with ‘~(~A & ~B)’, the leaves with ‘A’ and ‘B’ respectively, and the other nodes with the appropriate formulae.

According to our formation rules, any application of a rule for a *binary* connective produces a string of symbols with outer parentheses around it. This means that, strictly speaking, few of the formulae displayed on preceding pages which are not negations are well formed, since most of them lack outer parentheses. For example, though '(~A & ~B)' is a wff, '~A & ~B' is not. But outer parentheses do no work in determining the meaning of a string of symbols, and displaying them merely increases clutter. So we assume the *invisibility* convention for outer parentheses on formulae displayed on the page: the parentheses are really there, it is just that they are invisible. Hence, '~A & ~B' is a wff after all. Of course, if we embed '~A & ~B' as a subformula in a more complex formula, the invisible parentheses become visible again, as illustrated for example by 'C & ~(~A & ~B)'. It is only outer parentheses for a complete formula which may be invisible.

Here is the parse tree for formula (3.s) of §4, 'X → [~(~J ∨ (~M ∨ ~S)) → ~V]'. (We regard brackets as a permissible notational variant of parentheses.)



Choosing sentence-letters for the leaves of the tree, at the first stage in the construction of (4.3.s), is justified by the rule (*f-sl*). As in the previous example, we have used invisible outer parentheses at the root node, while at each intermediate node not generated by an application of (*f*-~), the formula formed at that node has visible outer parentheses. It is the use of these parentheses at intermediate nodes, together with the scope convention for '~', which guarantees that each string of symbols produced by the formation rules can be read in one and only one way. Note also that in arriving at (4.3.s), there was a point at which we made an arbitrary choice, concerning how to group the disjuncts of '~J ∨ ~M ∨ ~S' to make a two-disjunct disjunction. We chose the grouping '~J ∨ (~M ∨ ~S)', though '(~J ∨ ~M) ∨ ~S' would have served equally well. Had we made the latter choice, the difference in the parse tree would be that '~J' and '~M' would be grouped by a first application of (*f*-∨), and then the result would be grouped with '~S' by a second application of (*f*-∨).

Parse trees can be read top down or bottom up. Reading them top down, we see how to break the formula into its syntactic components. Reading bottom up, we see how the formula can be built up from its atomic constituents. However, in actually constructing a parse tree, it is usually easier to work from the top down.

We can use parse trees or the formation rules to distinguish well-formed formulae from non-well-formed ones. Although some strings of symbols are obviously not well-formed, such as '~& B C', there are other non-well-formed strings with which the problem is more subtle. For example, in 'A → ((~B) & C)' the flaw is with the parentheses surrounding '~B'; the formation rule (*f*-~) does not put parentheses around the negative formula formed by prefixing '~', so the parentheses in this example are incorrect. Non-well-formed strings like this can be diagnosed only by paying close attention to the exact requirements of the formation rules.

We are now in a position to give precise definitions of such concepts as 'main connective', 'scope' and their ilk. We begin with a preliminary definition, that of a *subformula* of a formula.

- If *p* is any wff, *q* is a *subformula* of *p* if and only if *q* labels some node of the parse tree for *p*.

By this definition, everything from the sentence-letters in *p* to *p* itself is a subformula of *p*. So we can list the subformulae of '~J ∨ (~M ∨ ~S)' as: 'J', 'M', 'S', '~J', '~M', '~S', '(~M ∨ ~S)', and '~J ∨ (~M ∨ ~S)'. The other definitions we promised can now be stated:

- If *p* is any wff and a binary connective *c* occurs in *p*, then the *scope* of that occurrence of *c* in *p* is the subformula of *p* which appears at the node in the parse tree of *p* at which that occurrence of *c* is introduced. The scope of an occurrence of '~' is the subformula at the node *immediately preceding* the one where that '~' is introduced.
- If *p* is any wff, the *main connective* of *p* is the connective introduced at the root node of *p*'s parse tree. If the main connective of *p* is '~', the formula to which the '~' is prefixed is the *main subformula* of *p*. If the main connective of *p* is a binary connective, then the subformulae which it connects are the *main subformulae* of *p*.
- If *c* and *c′* are two occurrences of connectives in *p*, then *c′* is *within the scope of c* if and only if *c′* occurs in the subformula which is the scope of *c*.

These definitions give us precisely the right results. For example, the main connective of (3.s) on page 30 is the leftmost '→' which occurs in it, while the scope of the leftmost '~' is '(~J ∨ (~M ∨ ~S))'. But the rightmost '~' in (3.s) is not within the scope of the leftmost, since the rightmost '~' does not occur in the formula at the node preceding the one where the leftmost '~' is introduced. One slightly odd consequence of the definitions which is perhaps worth mentioning is that when a negation has within its scope a formula whose own main connective is binary, the subformula which is the scope of the negation and the subformula which is the scope of that binary connective are the same.

❑ Exercises

I Write parse trees for the following LSL wffs:

(1) (~P & Q) → R
(2) ~(P & Q) → R
(3) ~[(P & Q) → R]
(4) A ↔ (~B → (C ∨ ~C))

((4) requires that 'C' be listed twice, at the bottom of two paths in the tree)

(a) List the subformulae of (4).
(b) What is the scope of the leftmost '~' in (4)?
(c) What is the scope of '→' in (4)?
(d) In (1), is '~' within the scope of '&' or vice versa?
*(e) Which is within the scope of which in (2) and (3)?
(f) In (4), which connective(s) is the leftmost '~' within the scope of?
(g) What about the other '~' in (4)?

II Which of the following are not wffs of LSL? For each symbol-string you identify as a non-wff, briefly explain why it is impossible to assemble it with the formation rules (review the discussion of 'A → ((~B) & C)' on page 39).

(1) ((~P & Q) → R)
(2) (A ↔ ((B ↔ C)))
(3) (P → ~Q & R)
*(4) (~(~P & ~Q))
(5) (~S → (~T ↔ (U & (V ∨ W)))

6 Quotation, selective and otherwise

The remaining grammatical issue we have to deal with concerns the proper use of quotation marks, which to this point we have been using rather loosely. In English, a central use of quotation marks is to allow us to talk about words or symbols rather than the things for which they stand. For instance, we can contrast (1) and (2):

(1) Paris is the capital of France.

(2) 'Paris' contains five letters.

Without the quotation marks, (2) would make little sense, since it would say that Paris-the-city contains five letters. The distinction exhibited by (1) and (2) is called the *use/mention* distinction: in (1) we *use* the word 'Paris' to make a statement about the city it stands for, while in (2) we *mention* the word 'Paris' to say something about it (the word). Thus on previous pages, when we have wanted to say something about, that is, *mention*, particular formulae of LSL, we have put those formulae in quotes.

This works well enough so long as we only want to mention individual formulae, as in

(3) 'A & B' is a conjunction.

But ordinary quotes do not work if we want to make generalizations. Suppose we want to say that conjoining any two LSL wffs yields an LSL wff. Then we might try

(4) If *p* and *q* are LSL wffs, so is '(*p* & *q*)'.

But this is incorrect: (4) is like

(5) If *x* and *y* are even numbers, so is '(*x* + *y*)'.

'(*x* + *y*)' is not a number at all, but simply a string of symbols. Similarly, '(*p* & *q*)' is not an LSL wff, since '*p*' and '*q*' are not LSL sentence-letters (in the specification of the lexicon of LSL on page 35, sentence-letters were all defined to be uppercase roman). However, if the quotation marks are removed from (5), all is well; not so with (4), for in (4) we *are* trying to mention formulae, though not any *specific* formula. Some kind of quotation is still involved. What we are really trying to say with (4) is the following:

(6) If *p* and *q* are LSL wffs, so is the symbol-string which results from writing '(' followed by *p* followed by '&' followed by *q* followed by ')'.

which is exactly how we *did* say it in the syntax clause (*f*-&) on page 36. In these syntax clauses, and in (6), we use quotation marks *selectively*, putting them around the parentheses and the connectives but not around the sentential variables '*p*' and '*q*'.

The rationale for selective quotation is best understood in the light of the distinction between the *object language* and the *metalanguage*. In a given context, the object language is the language under discussion, and the metalanguage is the language in which the discussion is being carried out. Thus in the context of a French class, French is the object language and the metalanguage is usually the native language of the students (though in 'immersion' courses, French will be the metalanguage as well). At this point in this book, the object language is LSL and the metalanguage is the language in which the book is written, English augmented by some technical notation. Sometimes in discussing an object language we want to say things not about a particular object-language sentence but about a class of sentences characterized syntactically. The characterization may involve elements of the object language and of the metalanguage, as in (6), where we want to talk about the class of conjunctions in LSL. The letters '*p*' and '*q*' which we use to stand for arbitrary symbol-strings or wffs of LSL do not themselves belong to LSL. They are part of the metalanguage, augmented English, and for that reason are known as *metavariables* for sentences. But the symbol '&' is an LSL symbol, so it is something we mention, rather than use, in the metalanguage—throughout this book, whenever we want to use an expression for a connective in the metalanguage, we will use English word(s), thereby keeping object language and metalanguage apart. What should be used and what should be mentioned in the present context is made clear in (6), where ordinary quotes surround everything being mentioned. The trouble with (4) is that the ordinary quotes have the effect of creating a mention of the whole expression '(*p* & *q*)', where all we really want to mention are the conjunction symbol and the parentheses, that is, the items from the object language.

Rather than write out clauses like (6) every time we say something about a general class of formulae, we can abbreviate by using *corner quotes*, also called *selective* quotes. Corner quotes around a whole expression put ordinary quotes selectively around parts of that expression, those parts which are object language symbols. The corners simply abbreviate fully explicit formulations such as (6). Thus the shorter version of (6) is:

(7) If *p* and *q* are LSL wffs, so is $\ulcorner(p \ \& \ q)\urcorner$.

The syntax clauses given on page 36 may themselves be abbreviated by clauses in the style of (7). Thus instead of writing

(*f*-↔) For any symbol-strings *p* and *q*, if *p* and *q* are wffs, then the result of writing '(' followed by *p* followed by '↔' followed by *q* followed by ')' is a wff

we could have written

(8) (*f*-↔): For any symbol-strings *p* and *q*, if *p* and *q* are wffs then $\ulcorner(p \leftrightarrow q)\urcorner$ is a wff.

The other syntax clauses may be abbreviated in the same way. We can contrast the effect of corners and ordinary quotes with an example. Suppose *p* is the LSL sentence '(A & B)', and *q* is the LSL sentence 'C'. Then the result of writing *p* followed by '↔' followed by *q* is '(A & B) ↔ C'; that is, $\ulcorner p \leftrightarrow q\urcorner$ = '(A & B) ↔ C'. But the result of writing '*p*' followed by '↔' followed by '*q*' is just '*p* ↔ *q*'; the fact that '*p*' and '*q*' are metalanguage symbols which stand for LSL wffs is irrelevant when we mention them by putting ordinary quotes around them. So if we want to use certain metalanguage symbols to stand for pieces of the object language, corners must be employed.

❑ Exercises

I Rewrite the formation rules (*f*-~) and (*f*-→) in abbreviated form using corners (compare (7) of this section).

II (1) Let *p* be the English phrase 'snow is' and let *q* be the word 'white'.

- *(a) Write out the result of writing *p* followed by *q*, that is, write out ⌜*p q*⌝.
- *(b) Write out the result of writing *p* followed by '*q'*.

(2) Let *p* be '(A & B)' and let *q* be 'R'.

- (a) Write out ⌜*p* ∨ *q*⌝.
- (b) Write out '*p*' followed by *q*.

III On page 21 and page 23 rules for symbolizing sentences with 'only if' and 'unless' are stated, but strictly speaking incorrectly. Use corners to state them correctly.

IV Which of the following statements contain misuses of English quotation marks or corner quotes, or omissions where use is required? For each such statement which you identify, explain briefly what is wrong with it and how it could be made correct.

- *(1) Rome is the name of a city.
- (2) 'Rome' is the largest city in Italy.
- (3) Superman's other name is Clark Kent.
- *(4) If *p* is an indicative English sentence, ⌜it is not the case that *p*⌝ is also an indicative English sentence.
- (5) If *p* is an indicative sentence of French, so is ⌜*p*, n'est ce pas?⌝.
- (6) If *p* is a wff of LSL so is ⌜~*p*⌝.
- (7) If ⌜(*p* & *q*)⌝ is a true conjunction of LSL then *p* is true and *q* is true.
- (8) In French, Vive and la and France can go together to form ⌜Vive la France⌝.
- (9) In LSL, A and B can go together to form ⌜A & B⌝.
- (10) If '*p'* is a complete sentence of English then it contains a subject and a verb.
- *(11) If *p* is an indicative sentence of English then it is obvious to every English speaker that *p* is an indicative sentence of English.

(Note: Allowing for syntactic error, (11) is ambiguous—on one reading it is syntactically correct but for the other requires quotes. Explain.)

V In which of the statements under IV is English both the object language and the metalanguage? (You may count corners and the use of metavariables for sentences as part of English.) In IV.5, what is the object language and what is the metalanguage?

7 Summary

- Many connectives such as 'only if' and 'unless' can be expressed in terms of our basic five.
- To symbolize an argument, we identify its premises and conclusion and the atomic sentences which occur in it. We then create a dictionary and substitute letters for atomic sentences as an intermediate step.
- The resulting formulae belong to a language, LSL, with its own syntax.
- The syntactic rules determine parse trees for well-formed formulae, that is, sentences of LSL, and syntactic concepts like 'main connective' can be defined using parse trees.
- We use corners around phrases which characterize classes of object language expressions and which contain metalanguage symbols being used and object language symbols (usually connectives) being mentioned.