# Solutions to Selected Exercises

## Chapter 2, §2

(6) I: Inflation is falling
G: The government can guide the economy wisely
R: The government can regain its popularity

$$I \,\&\, {\sim}(G \,\&\, R)$$

[“can’t _ and at the same time _” is the denial of a conjunction: not both are true. So we should have ‘~(_ & _)’.]

(9)

| | |
|---|---|
| S: Smith is a lawyer | B: Brown is a lawyer |
| R: Robinson is a lawyer | X: Smith is honest |
| Y: Brown is honest | Z: Robinson is honest |

$$(S \,\&\, (B \,\&\, R)) \,\&\, ((X \,\&\, Y) \vee ((Y \,\&\, Z) \vee (X \,\&\, Z)))$$

[‘Still’ is another way of expressing conjunction. ‘At least two of them are honest’ should be expanded into full sentential form as a claim explicitly about Smith, Brown and Robinson. If at least two are honest, then of the three, either the first two are honest (‘X & Y’), or the second two are (‘Y & Z’), or the first and third are (‘X & Z’).]

## Chapter 2, §3

(5) P: Grades may be posted
R: Students request grades to be posted
S: Students are identified by name

$$(R \,\&\, {\sim}S) \rightarrow P$$

(To say that a condition p will hold provided q holds is to say that *q* is sufficient for p, as in ‘you’ll be notified provided you’re registered’—this implies that there is nothing else you must do. But it leaves it open that there are ways an unregistered person may be notified, say by some special arrangement.)

(8) R: One has the right to run for president
C: One is an American citizen

$R \leftrightarrow C$

(12) E: One makes an effort
P: One passes the exam

$E \rightarrow P$

(15) F: There is an increase in government funding for education
Q: There is an improvement in the quality of education

$(Q \rightarrow F) \,\&\, {\sim}(F \rightarrow Q)$

## Chapter 2, §4

(4) H: Homer existed
C: The Odyssey was written by a committee
B: Butler was right
W: The Odyssey was written by a woman

${\sim}H \rightarrow (C \vee (B \rightarrow W)), {\sim}W \therefore C$

(9) P: Parking is prohibited in the center of town
B: People will buy bicycles
O: The general inconvenience is offset
U: City shops pick up some business

$P \rightarrow B, B \rightarrow (O \,\&\, U), P \rightarrow {\sim}U \therefore {\sim}B$

['In which case' means 'in the case that people buy bicycles'. The best we can do for 'the general inconvenience is offset by the fact that…' is to say that the general inconvenience is offset *and* shops pick up some business.]

## Chapter 2, §5

(I.e) In (2) the '&' is within the scope of the '~', since the scope of the '~', which by definition is the formula at the node in the parse tree for (2) immediately beneath the node where the '~' is introduced, is '(P & Q)', which contains the '&' in question. In (3) the '&' is again within the scope of the '~', since in the parse tree for (3), the '&' appears in the formula which is the scope of the '~', the formula at the node where the '~' is introduced (the root node).

(II.4) The outer parentheses in (4) are incorrect, since in its structure the last step (reading bottom to top) is to apply the formation rule (f-~), and this rule (page 36) does not put outer parentheses around the formula which results from its application.

## Chapter 2, §6

(II.1) (a) If p = 'snow is' and q = 'white', then ⌜p q⌝ (note the space) is 'snow is white'. (b) The result of writing p followed by 'q' is 'snow isq' or, with optional space, 'snow is q'.

(IV.1) This is incorrect—what is meant is that 'Rome' is the name of a city. Rome is a city, not a city's name.

(IV.4) This is grammatically correct as it stands. It is also true: if p is an indicative sentence of English, so is the result of prefixing 'it is not the case that' to it.

(IV.11) The reading on which the statement is syntactically incorrect is the one on which what is meant is that if we prefix 'it is obvious to every English speaker that' to any syntactically correct indicative sentence we obtain a syntactically correct English sentence. This claim, which is in fact true, requires corners around 'it is obvious to every English speaker that p'. On the other reading (can you see what it is?) the claim is syntactically correct but false.

## Chapter 3, §1

(4) If A is a Knight, then since that means what he says is true, B is a Knave and so speaks falsely. Thus A and C are of *different* types, and therefore C is a Knave. This is a possible solution, but for it to be *determined* what C is, there must be no solution on which C is a Knight. So what happens if A is a Knave? Then B is a Knight since A speaks falsely, so A and C are of the same type, since B speaks truly, being a Knight, so again C is a Knave. Thus being a Knave is the only possibility for C.

## Chapter 3, §2

(I.4)

| A B | (A ↔ B) | & | (A & ~B) |
|---|---|---|---|
| ⊤ ⊤ | ⊤ | $\boxed{\bot}$ | ⊥ |
| ⊤ ⊥ | ⊥ | $\boxed{\bot}$ | ⊤ |
| ⊥ ⊤ | ⊥ | $\boxed{\bot}$ | ⊥ |
| ⊥ ⊥ | ⊤ | $\boxed{\bot}$ | ⊥ |

Since the final column of the table for formula (1d) contains only ⊥s, (1d) is a contradiction.

## Chapter 3, §4

(I.4) There are two cases, (1) 'A' is true, 'C' is false, and (2) the converse. In case 1, premise 1 false, so only case 2 need be considered. This requires 'D' to be true for premise 3 to be true and hence 'B' must be true for premise 2 to be true. This interpretation makes all premises true and the conclusion false, so the argument-form is invalid, as demonstrated by the interpretation assigning $\perp$ to 'A' and $\top$ to 'B', 'C' and 'D'.

(II.4) F: Yossarian flies his missions
D: Yossarian puts himself in danger
R: Yossarian is rational
A: Yossarian asks to be grounded

$(F \rightarrow D) \ \& \ (D \rightarrow {\sim}R), (R \rightarrow A) \ \& \ ({\sim}F \rightarrow A), ({\sim}F \rightarrow {\sim}R) \ \& \ (A \rightarrow R)$
$\therefore (R \vee {\sim}R) \rightarrow F$

['Only irrational people are grounded' particularized to Yossarian means 'Yossarian will be grounded (i.e. doesn't fly his missions) only if Yossarian is not rational'; 'a request to be grounded is proof of rationality' particularized to Yossarian means 'if Yossarian asks to be grounded then Yossarian is rational'.]

To test for validity, we note that 'F' must be false for the conclusion to be false; hence 'A' must be true for premise 2 to be true, and so 'R' must be true for the second conjunct of premise 3 to be true; but 'R' has to be false for the first conjunct to be true. Thus the argument-form is valid.

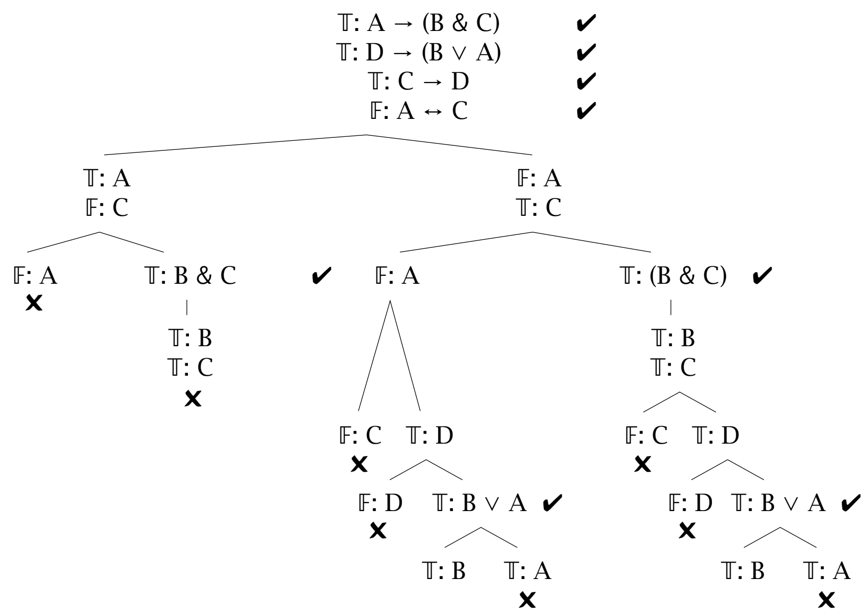
## Chapter 3, §5

(I.4) $A \rightarrow (B \ \& \ C), D \rightarrow (B \vee A), C \rightarrow D \vDash A \leftrightarrow C$

The solution is displayed on the next page. In this inverted tree there are ten paths, eight of which are closed. The two open paths, the fifth and ninth (reading along the leaves from the left), determine the same interpretation, the one assigning $\perp$ to 'A' and $\top$ to 'B', 'C' and 'D'. So this result agrees with the one obtained by the method of constructing an interpretation.

## Chapter 3, §6

(3) If $p \vDash (q \ \& \ r)$, then on no assignment do we have p true, $(q \ \& \ r)$ false; so none of (a), p is true, q is true, r is false, (b) p is true, q is false, r is true, (c) p is true, q is false, r is false, is possible. If $(p \leftrightarrow q) \nvDash (p \leftrightarrow r)$, then there are assignments of truth-values to sentence-letters in *p*, q and r on which $(p \leftrightarrow q)$ is true, $(p \leftrightarrow r)$ is false. This is possible, since, comparing (a), (b) and (c), we see that we have not ruled out an assignment on which e.g. p is false, q is false, r is true. So it does not follow that $(p \leftrightarrow q) \vDash (p \leftrightarrow r)$.




In this inverted tree there are ten paths, eight of which are closed. The two open paths, the fifth and ninth reading along the leaves from the left, determine the same interpretation, the one assigning ⊥ to 'A' and ⊤ to 'B', 'C' and 'D'. So this result agrees with the one obtained by the method of constructing an interpretation.

## Chapter 3, §6 (continued)

(8) Semantic consequence holds when no interpretation makes the premises true and the conclusion false. Consequently, if no interpretation at all makes the conclusion false, then no interpretation makes the premises true and the conclusion false, and so semantic consequence holds no matter what the premises are. Thus every semantic sequent with a tautology (such as '(A ∨ ~A)') as its conclusion is correct.

## Chapter 3, §7

(I) The formula in {~,&,∨} for table 2 is:

$$(A \mathbin{\&} {\sim}B \mathbin{\&} C) \vee (A \mathbin{\&} {\sim}B \mathbin{\&} {\sim}C) \vee ({\sim}A \mathbin{\&} {\sim}B \mathbin{\&} C) \vee ({\sim}A \mathbin{\&} {\sim}B \mathbin{\&} {\sim}C).$$

To eliminate all occurrences of '∨' we have to apply the substitution-rule given on page 79 three times. Eliminating the first occurrence of '∨' yields:

~[~(A & ~B & C) & ~[(A & ~B & ~C) ∨ (~A & ~B & C) ∨ (~A & ~B & ~C)]].

Here we put ‘(A & ~B & C)’ for r and ‘(A & ~B & ~C) ∨ (~A & ~B & C) ∨ (~A & ~B & ~C)’ for s in the substitution rule. Eliminating the first occurrence of ‘∨’ in the latter formula yields

~[~(A & ~B & C) & ~{~[(A & ~B & ~C) & ~[(~A & ~B & C) ∨ (~A & ~B & ~C)]]}].

Finally, we eliminate the remaining ‘∨’ to obtain

~[~(A & ~B & C) & ~{~[(A & ~B & ~C) & ~[~{~(~A & ~B & C) & ~(~A & ~B & ~C)}]]}].

In this last step, the subformula ⌜r ∨ s⌝ for which substitution has been made is ‘(~A & ~B & C) ∨ (~A & ~B & ~C)’.

(II.3) ⌜p ← q⌝ has the table ⊤ ⊤ ⊥ ⊤. Since {~,&,∨} is functionally complete, then given any truth-table, we can find a formula p in {~,&,∨} which has that table. We need two substitution rules, one of which allows us to replace the occurrences of ‘&’ in p (if any), the other of which allows us to replace the occurrences of ‘∨’ (if any). Since the rules will replace formulae with logically equivalent formulae, the result will be a formula in {~,←} which also has the given truth-table. The following rules are correct:

(1) Replace every subformula of p of the form ⌜r & s⌝ with ⌜~(~r ← s)⌝.
(2) Replace every subformula of p of the form ⌜r ∨ s⌝ with ⌜r ← ~s⌝.

(3) The simplest solution is to show that, using just ‘~’ and ‘↔’, there is no way of expressing a two-place truth-function which over the four possible inputs has exactly one or exactly three ⊤s in its output. In other words, no formula in ‘~’ and ‘↔’ with two sentence-letters p and q has an odd number of ⊤s in its four-row truth-table. First we establish the ***Minor Lemma***, that if any formula has an even number of ⊤s in its table, so does its negation. *Proof:* The number of rows in any truth-table is even, so a formula with an even number of ⊤s also has an even number of ⊥s, producing an even number of ⊤s for the formula’s negation.

Next, we prove the ***Major Lemma***, that if two columns L and R in a four-row truth-table each have an even number of ⊤s, so does the result of applying the biconditional truth-function to these columns. *Proof:* Assume that L and R each have an even number of ⊤s. Then there are three possibilities, (1) no ⊤s in L, (2) two ⊤s in L, (3) four ⊤s in L. Each of cases (1), (2) and (3) subdivides into three further cases (a), (b) and (c). *Case 1:* (a) No ⊤s in R. Then each column has four ⊥s, so the resulting table has four ⊤s; (b) Two ⊤s in R. For Case 1 this means two rows have the same values and two rows have different values; hence the resulting table has two ⊤s; (c) Four ⊤s in R. Then the resulting table has no ⊤s. *Case 2:* (a) No ⊤s in R. This is essentially the same as Case 1b; (b) Two ⊤s in R. If these ⊤s face the two ⊤s in L then there are four ⊤s in the table. If neither faces a ⊤ in L then there are no ⊤s in the table. If just one faces a ⊤

in L then two rows have the same value and two rows have different values, so there are two ⊤s in the table; *Case 3:* (a) No ⊤s in R. Essentially the same as Case 1c; (b) Two ⊤s in R. See Case 2c; (c) Four ⊤s in R. Then all rows have ⊤s on them and so the table has four ⊤s as well. This proves the Major Lemma. (The argument is a special case of a more general argument that can be given for any number $2^n$ of rows.)

Now let B be an arbitrary formula in p, q, '~' and '↔'. We show that B's 4-row truth-table contains an even number of ⊤s (and hence of ⊥s) in its final column. (a) Suppose B contains no occurrences of '↔'. Then B is either a sentence-letter, in which case its column contains an even number of ⊤s, or a sentence-letter prefixed by one or more '~'s, in which case, by repeated applications of the Minor Lemma, it still contains an even number of ⊤s. (b) Suppose B contains one occurrence of '↔'. Then in case (b1), B is a biconditional with two sides, each of which contains no occurrence of '↔', and so by (a), each side of B has an even number of ⊤s in its table. Hence by the Major Lemma, B has an even number of ⊤s in its table. In case (b2), B consists in a biconditional C prefixed with one or more occurrences of '~'; by (b1), C has an even number of ⊤s in its table, so by repeated applications of the Minor Lemma, B has an even number of ⊤s in its table. (c) Suppose B contains two occurrences of '↔'. Then in case (c1), B is a biconditional with two sides, each of which contains one or no occurrence of '↔', and so by (a) and (b), each side of B has an even number of ⊤s in its table. Hence by the Major Lemma, B has an even number of ⊤s in its table. In case (c2), B consists in a biconditional C prefixed with one or more occurrences of '~'; by (c1), C has an even number of ⊤s in its table, so by repeated applications of the Minor Lemma, B has an even number of ⊤s in its table.

Continuing in this way we can show that for any n, if B is a formula in two sentence-letters p, q, '~' and '↔' with n occurrences of '↔', B has an even number of ⊤s in its table (the reader who is familiar with strong mathematical induction should think about how we could go about making 'continuing in this way' rigorous). Hence {~,↔} is expressively incomplete; for example, '→' is undefinable.

## Chapter 3, §8

(3) (i) Let 'A' mean 'the speed of light does not vary with the motion of its source'. Then 'A' is true and also 'It is surprising that A' is true. (ii) Let 'A' mean 'there are more robberies when the police are on strike'. Then 'A' is true but 'It is surprising that A' is false. Thus no entry can be made in the top row of a purported function-table for 'It is surprising that'.

(5) (i) Let 'A' mean 'lead sinks in water' and 'B' mean 'lead is denser than water'. Then 'A' and 'B' are both true, and in addition, 'A, which means that B' is true. (ii) Let 'A' mean 'lead sinks in water' and 'B' = 'Moses wrote the Pentateuch'. Then 'A' and 'B' are true, but 'A, which means that B' is false. So no entry can be made in the top row of a purported function-table for '…, which means that…'.

## Chapter 4, §2

| | | | | |
|---|---|---|---|---|
| (2) | 1 | (1) | $A \rightarrow B$ | Premise |
| | 2 | (2) | $A \rightarrow C$ | Premise |
| | 3 | (3) | $A$ | Assumption |
| | 1,3 | (4) | $B$ | 1,3 →E |
| | 2,3 | (5) | $C$ | 2,3 →E |
| | 1,2,3 | (6) | $B \& C$ | 4,5 &I |
| | 1,2 | (7) | $A \rightarrow (B \& C)$ | 3,6 →I ◆ |

| | | | | |
|---|---|---|---|---|
| (5) | 1 | (1) | $(A \& B) \rightarrow C$ | Premise |
| | 2 | (2) | $A$ | Assumption |
| | 3 | (3) | $B$ | Assumption |
| | 2,3 | (4) | $A \& B$ | 2,3 &I |
| | 1,2,3 | (5) | $C$ | 1,4 →E |
| | 1,2 | (6) | $B \rightarrow C$ | 3,5 →I |
| | 1 | (7) | $A \rightarrow (B \rightarrow C)$ | 2,6 →I ◆ |

| | | | | |
|---|---|---|---|---|
| (16) | 1 | (1) | $A \rightarrow B$ | Premise |
| | 2 | (2) | $A$ | Assumption |
| | 3 | (3) | $C$ | Assumption |
| | 1,2 | (4) | $B$ | 1,2 →E |
| | 1,2 | (5) | $C \rightarrow B$ | 3,4 →I |
| | 1 | (6) | $A \rightarrow (C \rightarrow B)$ | 2,5 →I ◆ |

## Chapter 4, §3

| | | | | |
|---|---|---|---|---|
| (4) | 1 | (1) | $A \rightarrow (B \& C)$ | Assumption |
| | 2 | (2) | $A$ | Assumption |
| | 1,2 | (3) | $B \& C$ | 1,2 →E |
| | 1,2 | (4) | $B$ | 3 &E |
| | 1 | (5) | $A \rightarrow B$ | 2,4 →I |
| | 1,2 | (6) | $C$ | 3 &E |
| | 1 | (7) | $A \rightarrow C$ | 2,6 →I |
| | 1 | (8) | $(A \rightarrow B) \& (A \rightarrow C)$ | 5,7 &I |
| | | (9) | $[A \rightarrow (B \& C)] \rightarrow [(A \rightarrow B) \& (A \rightarrow C)]$ | 1,8 →I ◆ |

## Chapter 4, §4

| | | | | |
|---|---|---|---|---|
| (1) | 1 | (1) | $A \rightarrow \sim B$ | Premise |
| | 2 | (2) | $B$ | Assumption |
| | 3 | (3) | $A$ | Assumption |
| | 1,3 | (4) | $\sim B$ | 1,3 →E |
| | 1,2,3 | (5) | $\curlywedge$ | 4,2 ~E |
| | 1,2 | (6) | $\sim A$ | 3,5 ~I |
| | 1 | (7) | $B \rightarrow \sim A$ | 2,6 →I ◆ |

| | | | | |
|---|---|---|---|---|
| (8) | 1 | (1) | A & ~A | Assumption |
| | 2 | (2) | ~B | Assumption |
| | 1 | (3) | A | 1 &E |
| | 1 | (4) | ~A | 1 &E |
| | 1 | (5) | ⋏ | 3,4 ~E |
| | 1 | (6) | ~~B | 2,5 ~I |
| | 1 | (7) | B | 6 DN |
| | | (8) | (A & ~A) → B | 1,7 →I ◆ |

| | | | | |
|---|---|---|---|---|
| (13) | 1 | (1) | A → B | Premise |
| | 2 | (2) | B → ~A | Premise |
| | 3 | (3) | A | Assumption |
| | 1,3 | (4) | B | 1,3 →E |
| | 1,2,3 | (5) | ~A | 2,4 →E |
| | 1,2,3 | (6) | ⋏ | 5,3 ~E |
| | 1,2 | (7) | ~A | 3,6 ~I ◆ |

| | | | | |
|---|---|---|---|---|
| (18) | 1 | (1) | (A & ~B) → ~A | Premise |
| | 2 | (2) | A | Assumption |
| | 3 | (3) | ~B | Assumption |
| | 2,3 | (4) | A & ~B | 2,3 &I |
| | 1,2,3 | (5) | ~A | 1,4 →E |
| | 1,2,3 | (6) | ⋏ | 5,2 ~E |
| | 1,2 | (7) | ~~B | 3,6 ~I |
| | 1,2 | (8) | B | 7 DN |
| | 1 | (9) | A → B | 2,8 →I ◆ |

## Chapter 4, §5

| | | | | |
|---|---|---|---|---|
| (4) | 1 | (1) | A ∨ ~~B | Premise |
| | 2 | (2) | A | Assumption |
| | 2 | (3) | A ∨ B | 2 ∨I |
| | 4 | (4) | ~~B | Assumption |
| | 4 | (5) | B | 4 DN |
| | 4 | (6) | A ∨ B | 5 ∨I |
| | 1 | (7) | A ∨ B | 1,2,3,4,6 ∨E ◆ |

| | | | | |
|---|---|---|---|---|
| (10) | 1 | (1) | A ∨ B | Premise |
| | 2 | (2) | A → B | Assumption |
| | 3 | (3) | A | Assumption |
| | 2,3 | (4) | B | 2,3 →E |
| | 5 | (5) | B | Assumption |
| | 1,2 | (6) | B | 1,3,4,5,5 ∨E |
| | 1 | (7) | (A → B) → B | 2,6 →I ◆ |

| | | | | |
|---|---|---|---|---|
| (13) | 1 | (1) | $A \vee B$ | Premise |
| | 2 | (2) | $\sim A \; \& \sim B$ | Assumption |
| | 3 | (3) | $A$ | Assumption |
| | 2 | (4) | $\sim A$ | 2 &E |
| | 2,3 | (5) | $\curlywedge$ | 4,3 ~E |
| | 6 | (6) | $B$ | Assumption |
| | 2 | (7) | $\sim B$ | 2 &E |
| | 2,6 | (8) | $\curlywedge$ | 7,6 ~E |
| | 1,2 | (9) | $\curlywedge$ | 1,3,5,6,8 ∨E |
| | 1 | (10) | $\sim(\sim A \; \& \sim B)$ | 2,9 ~I ◆ |

## Chapter 4, §6

| | | | | |
|---|---|---|---|---|
| (4) | 1 | (1) | $(A \vee B) \leftrightarrow A$ | Premise |
| | 2 | (2) | $B$ | Assumption |
| | 1 | (3) | $((A \vee B) \rightarrow A) \; \& \; (A \rightarrow (A \vee B))$ | 1 Df |
| | 1 | (4) | $(A \vee B) \rightarrow A$ | 3 &E |
| | 2 | (5) | $A \vee B$ | 2 ∨I |
| | 1,2 | (6) | $A$ | 4,5 →E |
| | 1 | (7) | $B \rightarrow A$ | 2,6 →I ◆ |

| | | | | |
|---|---|---|---|---|
| (10) | 1 | (1) | $(A \vee B) \vee C$ | Premise |
| | 2 | (2) | $B \leftrightarrow C$ | Premise |
| | 2 | (3) | $(B \rightarrow C) \; \& \; (C \rightarrow B)$ | Df |
| | 4 | (4) | $A \vee B$ | Assumption |
| | 5 | (5) | $A$ | Assumption |
| | 5 | (6) | $C \vee A$ | 5 ∨I |
| | 7 | (7) | $B$ | Assumption |
| | 2 | (8) | $B \rightarrow C$ | 3 &E |
| | 2,7 | (9) | $C$ | 8,7 →E |
| | 2,7 | (10) | $C \vee A$ | 9 ∨I |
| | 2,4 | (11) | $C \vee A$ | 4,5,6,7,10 ∨E |
| | 12 | (12) | $C$ | Assumption |
| | 12 | (13) | $C \vee A$ | 12 ∨I |
| | 1,2 | (14) | $C \vee A$ | 1,4,11,12,13 ∨E ◆ |

## Chapter 4, §8

(I.5) ‘$\sim(M \vee N) \vee (W \,\&\, U) \vdash_{NK} (M \vee N) \rightarrow (W \,\&\, U)$’ is a substitution-instance of (Imp): this sequent may be obtained from Imp by putting ‘$M \vee N$’ for ‘A’ and ‘$W \vee U$’ for ‘B’.

(II.iii) This sequent is a substitution-instance of (a). To obtain (iii) from (a), put ‘$\sim\sim(R \,\&\, S)$’ for ‘A’, ‘$\sim T \,\&\, S$’ for ‘B’, ‘$\sim\sim W$’ for ‘C’ and ‘$\sim T$’ for ‘D’.

| | | | | |
|---|---|---|---|---|
| (III.4) | 1 | (1) | $\sim B \rightarrow A$ | Premise |
| | 2 | (2) | $B \rightarrow A$ | Assumption |
| | | (3) | $B \vee \sim B$ | TI (LEM) |
| | 4 | (4) | $B$ | Assumption |
| | 2,4 | (5) | $A$ | 2,4 E |
| | 6 | (6) | $\sim B$ | Assumption |
| | 1,6 | (7) | $A$ | 1,6 $\rightarrow$E |
| | 1,2 | (8) | $A$ | 3,4,5,6,7 $\vee$E |
| | 1 | (9) | $(B \rightarrow A) \rightarrow A$ | 2,8 $\rightarrow$I ◆ |

| | | | | |
|---|---|---|---|---|
| (III.8) | 1 | (1) | $(A \vee B) \rightarrow (A \vee C)$ | Premise |
| | 2 | (2) | $\sim[A \vee (B \rightarrow C)]$ | Assumption |
| | 2 | (3) | $\sim A \,\&\, \sim(B \rightarrow C)$ | 2 SI (DeM) |
| | 2 | (4) | $\sim A$ | 3 &E |
| | 2 | (5) | $\sim(B \rightarrow C)$ | 3 &E |
| | 6 | (6) | $B$ | Assumption |
| | 6 | (7) | $A \vee B$ | 6 $\vee$I |
| | 1,6 | (8) | $A \vee C$ | 1,7 $\rightarrow$E |
| | 1,2,6 | (9) | $C$ | 8,4 SI (DS) |
| | 1,2 | (10) | $B \rightarrow C$ | 6,9 $\rightarrow$I |
| | 1,2 | (11) | $\curlywedge$ | 5,10 ~E |
| | 1 | (12) | $\sim\sim[A \vee (B \rightarrow C)]$ | 2,11 ~I |
| | 1 | (13) | $A \vee (B \rightarrow C)$ | 12 DN ◆ |

| | | | | |
|---|---|---|---|---|
| (III.13) | 1 | (1) | $(A \leftrightarrow B) \leftrightarrow (C \leftrightarrow D)$ | Premise |
| | 1 | (2) | $A \leftrightarrow [B \leftrightarrow (C \leftrightarrow D)]$ | 1 SI ((15), page 119) |
| | 1 | (3) | $\{A \rightarrow [B \leftrightarrow (C \leftrightarrow D)]\} \,\&\, \{[B \leftrightarrow (C \leftrightarrow D)] \rightarrow A\}$ | 2 Df |
| | 1 | (4) | $A \rightarrow [B \leftrightarrow (C \leftrightarrow D)]$ | 3 &E |
| | 1 | (5) | $[B \leftrightarrow (C \leftrightarrow D)] \rightarrow A$ | 3 &E |
| | 6 | (6) | $A$ | Assumption |
| | 1,6 | (7) | $B \leftrightarrow (C \leftrightarrow D)$ | 4,6 $\rightarrow$E |
| | 1,6 | (8) | $(B \leftrightarrow C) \leftrightarrow D$ | 7 SI ((15), page 119) |
| | 1,6 | (9) | $[(B \leftrightarrow C) \rightarrow D] \,\&\, [D \rightarrow (B \leftrightarrow C)]$ | 8 Df |
| | 1,6 | (10) | $(B \leftrightarrow C) \rightarrow D$ | 9 &E |
| | 1,6 | (11) | $D \rightarrow (B \leftrightarrow C)$ | 9 &E |
| | 12 | (12) | $D$ | Assumption |
| | 1,6,12 | (13) | $B \leftrightarrow C$ | 11,12 $\rightarrow$E |
| | 1,6,12 | (14) | $C \leftrightarrow B$ | 13 SI ((2), page 119) |
| | 1,6 | (15) | $D \rightarrow (C \leftrightarrow B)$ | 12,14 $\rightarrow$I |
| | 16 | (16) | $C \leftrightarrow B$ | Assumption |
| | 16 | (17) | $B \leftrightarrow C$ | 16 SI ((2), page 119) |
| | 1,6,16 | (18) | $D$ | 10,17 $\rightarrow$E |
| | 1,6 | (19) | $(C \leftrightarrow B) \rightarrow D$ | 16,18 $\rightarrow$I |
| | 1,6 | (20) | $[(C \leftrightarrow B) \rightarrow D] \,\&\, [D \rightarrow (C \leftrightarrow B)]$ | 19,15 &I |
| | 1,6 | (21) | $(C \leftrightarrow B) \leftrightarrow D$ | 20 Df |
| | 1,6 | (22) | $C \leftrightarrow (B \leftrightarrow D)$ | 21 SI ((15), page 119) |
| | 1 | (23) | $A \rightarrow [C \leftrightarrow (B \leftrightarrow D)$ | 6,22 $\rightarrow$I |

| | | | |
|---|---|---|---|
| 24 | (24) | C ↔ (B ↔ D) | Assumption |
| 24 | (25) | (C ↔ B) ↔ D | 24 SI ((15), page 119) |
| 26 | (26) | B ↔ C | Assumption |
| 26 | (27) | C ↔ B | 26 SI ((2), page 119) |
| 24 | (28) | [(C ↔ B) → D] & [D → (C ↔ B)] | 25 Df |
| 24 | (29) | (C ↔ B) → D | 28 &E |
| 24 | (30) | D → (C ↔ B) | 28 &E |
| 24,26 | (31) | D | 29,27 →E |
| 24 | (32) | (B ↔ C) → D | 26,31 →I |
| 33 | (33) | D | Assumption |
| 24,33 | (34) | C ↔ B | 30,33 →E |
| 24,33 | (35) | B ↔ C | 34 SI ((2), page 119) |
| 24 | (36) | D → (B ↔ C) | 33,35 →I |
| 24 | (37) | [(B ↔ C) → D] & [D → (B ↔ C)] | 32,36 &I |
| 24 | (38) | (B ↔ C) ↔ D | 37 Df |
| 24 | (39) | B ↔ (C ↔ D) | 38 SI ((15), page 119) |
| 1,24 | (40) | A | 5,39 →E |
| 1 | (41) | [C ↔ (B ↔ D)] → A | 24,40 →I |
| 1 | (42) | {A → [C ↔ (B ↔ D)]} & {[C ↔ (B ↔ D)] → A} | 23,41 &I |
| 1 | (43) | A ↔ [C ↔ (B ↔ D)] | 42 &E |
| 1 | (44) | (A ↔ C) ↔ (B ↔ D) | 43 SI (15, page 119) ◆ |

## Chapter 4, §9

(1) In the same style as the tree-format schema for ∨E on page 131, the schemata for the remaining rules are as follows:

$$\frac{p}{p \vee q}\ {\scriptstyle \vee I} \qquad \frac{p}{q \vee p}\ {\scriptstyle \vee I} \qquad \frac{\sim\sim p}{p}\ {\scriptstyle DN}$$

$$\frac{p \,\&\, q}{p}\ {\scriptstyle \&E} \qquad \frac{p \,\&\, q}{q}\ {\scriptstyle \&E} \qquad \frac{p \quad q}{p \,\&\, q}\ {\scriptstyle \&I}$$

$$\frac{p \qquad p \to q}{q}\ {\scriptstyle \to E} \qquad \frac{\sim p \qquad p}{\curlywedge}\ {\scriptstyle \sim E}$$

$$\begin{array}{c} \overline{[p]}^{\,(i)} \\ \vdots \\ \dfrac{q}{p \to q}\ {\scriptstyle \to I\ (i)} \end{array} \qquad \begin{array}{c} \overline{[p]}^{\,(i)} \\ \vdots \\ \dfrac{\curlywedge}{\sim p}\ {\scriptstyle \sim I\ (i)} \end{array}$$

In the rules →I and ~I, the notation indicates that we discharge every occurrence of p as leaf node on a path that contains the nodes labelled q and ⋏ respectively. Also, the rules permit but do not require the making and discharging of the assumption p. Thus the two-line proof A//B → A is a correct proof by →I in tree format of the sequent A $\vdash_{NK}$ B → A. One reason for this divergence from Lemmon format, in which the assumption 'B' would be displayed, is that in tree format there is no sensible place to put the assumption 'B', as some experimentation will reveal.

(2) A tree-format proof of Example 5.6, A ∨ (B ∨ C) $\vdash_{NK}$ (A ∨ B) ∨ C:

$$\dfrac{A \vee (B \vee C) \qquad \dfrac{\dfrac{\overline{A}^{\,(3)}}{A \vee B}{\scriptstyle \vee I}}{(A \vee B) \vee C}{\scriptstyle \vee I} \qquad \dfrac{\overline{B \vee C}^{\,(4)} \qquad \dfrac{\dfrac{\overline{B}^{\,(1)}}{A \vee B}{\scriptstyle \vee I}}{(A \vee B) \vee C}{\scriptstyle \vee I} \qquad \dfrac{\overline{C}^{\,(2)}}{(A \vee B) \vee C}{\scriptstyle \vee I}}{(A \vee B) \vee C}{\scriptstyle \vee E\,(1,2)}}{(A \vee B) \vee C}{\scriptstyle \vee E\,(3,4)}$$

## Chapter 4, §10

(3) First we show that every *S*-sequent is an NK-sequent. Suppose an *S*-proof contains a step of CR at line (m), and that this is the first application of CR. At line j we have the assumption ⌜~p⌝ and at line k we have '⋏', so in NK we may repeat the *S*-proof to line k and then by →I add the extra line ⌜~p → ⋏⌝, depending on $\{a_1,\ldots,a_n\}$/j. Then if we also have the sequent ~A → ⋏ $\vdash_{NK}$ A, we can apply SI to infer p in NK. To show ~A → ⋏ $\vdash_{NK}$ A:

| | | | |
|---|---|---|---|
| 1 | (1) | ~A → ⋏ | Premise |
| 2 | (2) | ~A | Assumption |
| 1,2 | (3) | ⋏ | 1,2 →E |
| 1 | (4) | ~~A | 2,3 ~I |
| 1 | (5) | A | 4 DN ◆ |

Every application of CR in an *S*-proof can be dealt with by the same combination of →I and SI; consequently, every sequent provable in *S* is also provable in NK. Conversely, we show that every sequent provable in NK is provable in *S*. NK proofs may use ~I and DN, which *S* lacks. But we can use SI in *S* to get the effect of an NK-application of DN, since we have ~~A $\vdash_S$ A.

| | | | |
|---|---|---|---|
| 1 | (1) | ~~A | Premise |
| 2 | (2) | ~A | Assumption |
| 1,2 | (3) | ⋏ | 1,2 ~E |
| 1 | (4) | A | 2,3 CR ◆ |

To get the effect of ~I in $S$ we show A → ⅄ $\vdash_S$ ~A. We already know ~~A $\vdash_S$ A, so we can use this sequent.

| | | | |
|---|---|---|---|
| 1 | (1) | A → ⅄ | Premise |
| 2 | (2) | ~~A | Assumption |
| 2 | (3) | A | 2, SI (~~A $\vdash_S$ A) |
| 1,2 | (4) | ⅄ | 1,3 →E |
| 1 | (5) | ~A | 2,4 CR ◆ |

(6) The difficult part is showing that every $S$-provable sequent is NK-provable, since Df$\sim_{Gen}$ can be applied to an arbitrary subformula of a formula on a line in an $S$-proof, and we need to provide a general explanation of why the same effect can always be obtained in NK. For example, in an $S$-proof we can move in one step from 'A ∨ (~(B & ~~C) → D)' to 'A ∨ (~(B & ~(C → ⅄)) → D)'. But what guarantee do we have in advance that there is an NK-derivation of 'A ∨ (~(B & ~(C → ⅄)) → D)' from 'A ∨ (~(B & ~~C) → D)'? The guarantee is provided by a procedure or algorithm $\mathcal{P}$ which can be applied to any formula φ to produce an NK-derivation of ψ from φ if ψ is $S$-derivable from φ by one application of Df$\sim_{Gen}$. The procedure is given as a collection of instructions, which may be applied to generate the next line of the derivation according to the main connective of the formula on the current line. If ψ is obtained from φ in $S$ by applying Df$\sim_{Gen}$ to the subformula ⌜~r⌝, we call ⌜~r⌝ the *target negation* and ⌜r → ⅄⌝ its transform. If ξ is a subformula of φ which contains the target negation ⌜~r⌝ we write ξ* for the result of replacing ⌜~r⌝ with ⌜r → ⅄⌝ in ξ. If ξ is a subformula of ψ which contains the target negation's transform ⌜r → ⅄⌝ we write ξ* for the result of replacing ⌜r → ⅄⌝ with ⌜~r⌝ in ξ (because of the way →I and ~I work, we may find ourselves working with formulae in which substitution of ⌜r → ⅄⌝ for ⌜~r⌝ has already been made). In understanding how the following procedure works the reader would be well-advised to apply it to 'A ∨ (~(B & ~~C) → D)' to see how it leads to a proof of 'A ∨ (~(B & ~(C → ⅄)) → D)'.

(&) If ⌜*p* & q⌝ is the formula at the current stage and *p* is the conjunct containing the target negation or its transform, apply &E to obtain *p* and q and continue to execute $\mathcal{P}$ on *p* until *p** is obtained; then apply &I to obtain ⌜*p** & q⌝. If q is the conjunct containing the target negation or its transform, apply &E to obtain *p* and q and continue to execute $\mathcal{P}$ on *q* until q* is obtained; then apply &I to obtain ⌜*p* & q*⌝.

(∨) If ⌜*p* ∨ q⌝ is the formula at the current stage and *p* is the disjunct containing the target negation or its transform, assume *p* and continue to execute $\mathcal{P}$ until *p** is obtained, then infer ⌜*p** ∨ q⌝ by ∨I; then assume q and immediately infer ⌜*p** ∨ q⌝ by ∨I. Finally, infer ⌜*p** ∨ q⌝ by ∨E. If q contains the target negation or its transform, proceed analogously to obtain ⌜*p* ∨ q*⌝ by ∨E.

(→) If ⌜*p* → q⌝ is the formula at the current stage and the antecedent *p* contains the target negation or its transform (as a proper subformula), assume *p** (note: *not p*) and continue to execute $\mathcal{P}$ until p is obtained. Then apply →E using ⌜*p* →

q⌝ to obtain q, then →I to obtain ⌜$p$* → q⌝. If ⌜$p$ → q⌝ is the formula at the current stage and the consequent q contains the target negation or its transform, assume $p$, derive q by →E, then continue with $\mathcal{P}$ until q* is obtained, and then apply →I to obtain ⌜$p$ → q*⌝.

(~) If ⌜~$p$⌝ is the formula at the current stage and $p$ contains the target negation (as a proper subformula) or its transform, then assume $p$* and continue to execute $\mathcal{P}$ until $p$ is obtained; then apply ~E using ⌜~$p$⌝ followed by ~I, thereby rejecting $p$* to obtain ⌜~$p$*⌝.

(SI) If none of the previous instructions apply then the formula at the current stage is the target negation or its transform. In this case we apply SI using ~A $\dashv\vdash_{NK}$ A → ⋏.

Since the same instructions serve to produce a derivation of φ from ψ, we have shown that every $\mathcal{S}$-provable sequent is NK-provable.

## Chapter 4, §11

(4) Given $\Sigma \vdash_{NK}$ A, then Σ,~A $\vdash_{NK}$ A also, by definition of '$\vdash_{NK}$' ('$\Sigma \vdash_{NK} p$' requires only that $p$ depend on a subset of Σ, so we get Σ,~A $\vdash_{NK}$ A by using the subset Σ of Σ,~A). Obviously, Σ,~A $\vdash_{NK}$ ~A, hence by ~E, Σ,~A $\vdash_{NK}$ ⋏. Thus Σ,~A is inconsistent. Conversely, if Σ,~A $\vdash_{NK}$ ⋏, then by ~I, $\Sigma \vdash_{NK}$ ~~A, and hence by DN, $\Sigma \vdash_{NK}$ A.

## Chapter 5, §2

(3) 'It is not the case that there is at least one person x such that x is a mathematician and x is famous': '~(∃x)(Mx & Fx)'; where 'F_' is '_ is famous' and 'M_' is '_ is a mathematician'.

(12) 'There is at least one thing x such that x is polluted, x is a city and x is smoggy and there is at least one thing x such that x is a city and x is polluted and x is not smoggy': '(∃x)((Px & Cx) & Sx) & (∃x)((Px & Cx) & ~Sx)'; where 'C_' is '_ is a city', 'P_' is '_ is polluted' and 'S_' is '_ is smoggy'.

(15) 'If there is at least one person x such that x is wealthy and x is an economist, then there is at least one person x such that x is famous and x is a mathematician': '(∃x)(Wx & Ex) → (∃x)(Fx & Mx)'; symbols as in (3), plus 'W_' is '_ is wealthy' and 'E_' is '_ is an economist'.

## Chapter 5, §3

(3) 'For all x, if x is an expensive university then x is private': '(∀x)((Ux & Ex) → Px)'. The italics in the English rule out the readings 'for all x, if x is private and

expensive, then x is a university', which says that the only expensive private things are universities ('only private *universities* are expensive'), and 'for all x, if x is expensive then x is a private university', which says that the only expensive things are private universities ('only *private universities* are expensive').

(7) 'It is not the case that for all x, if x glitters then x is gold'. Or: 'there is at least one thing x such that x glitters and x is not gold'. So: '~(∀x)(Gx → Ox)', or '(∃x)(Gx & ~Ox)'. Note that '(∀x)(Gx → ~Ox)' is wrong, since it says that everything which glitters is not gold, i.e., nothing which glitters is gold, and this is not what the English saying means.

(12) 'For all x, if x is an elected politician, then x is corrupt'; or, 'there does not exist an x such that x is an elected politician and x is incorrupt'. '(∀x)((Ex & Px) → Cx)' or '~(∃x)((Ex & Px) & ~Cx)'.

(16) 'For all x, if x is a wealthy logician then x is a textbook author'; '(∀x)((Wx & Lx) → Tx)'.

## Chapter 5, §4

(I.5) This formula is not a wff since the formation rule f-~ does not introduce parentheses around the formula which is formed when this rule is applied. The formula should be: '(∃x)~(∃y)(Fx & ~Gy)'.

(II.3) '~' is within the scope of '&' since the '~' already occurs in the formula formed at the node where the '&' is introduced by f-& in the parse tree for (iii). In other words, the node where the '&' is introduced dominates the node where the '~' is introduced.

## Chapter 6, §1

(3) '(∃x)(Fx & Gx)' is false because 'Fa & Ga', 'Fb & Gb' and 'Fc & Gc' are all false.

(8) '(∀x)(Hx → (∃y)(Jx & Iy))' has three instances, 'Ha → (∃y)(Ja & Iy)', 'Hb → (∃y)(Jb & Iy)' and 'Ha → (∃y)(Jb & Iy)'. The first instance is true since its antecedent is true and its consequent, '(∃y)(Ja & Iy)', is true (since 'Ja & Ic' is true). But the second instance is false, since 'Hb' is true but '(∃y)(Jb & Iy)' is false, because 'Jb & Ia', 'Jb & Ib' and 'Jb & Ic' are all false, 'Jb' being false in each case.

## Chapter 6, §2

(I.4) D = {α}, Ext(F) = ∅, Ext(G) = ∅, Ext(H) = ∅. Then '(∀x)(Fx ∨ Gx) ∨ (∀x)(Fx ∨ Hx)' is false since '(∀x)(Fx ∨ Gx)' is false (because 'Fa ∨ Ga' is false) and '(∀x)(Fx ∨ Hx)' is false (because 'Fa ∨ Ha)' is false). '(∀x)((Fx & Gx) → Hx)' is true because '(Fa & Ga) → Ha' is true, since its antecedent is false.

(I.13) D = {α,β}, Ext(F) = {α,β}, Ext(G) = {α}, Ext(H) = {α}, Ext(J) = ∅. '(∃x)(Fx & Gx) → (∀x)(Hx → Jx)' is false because '(∃x)(Fx & Gx)' is true ('Fa & Ga' is true) while '(∀x)(Hx → Jx)' is false ('Ha → Ja' is false). '(∀x)(Fx → Gx) → (∀x)(Hx → Jx)' is true since '(∀x)(Fx → Gx)' is false, because 'Fb → Gb' is false.

(I.20) D = {α,β}, Ext(F) = {α}, Ext(G) = ∅. '(∀x)(Fx → (∃y)Gy)' is false because 'Fa → (∃y)Gy' is false, since 'Fa' is true and '(∃y)Gy' is false ('Ga' and 'Gb' are both false). '(∀x)Fx → (∃y)Gy' is true because '(∀x)Fx' is false, since 'Fb' is false.

(I.27) D = {α,β}, Ext(F) = {α}, Ext(G) = {α}. '(∀x)[(∃y)Gy → Fx]' is false because '(∃y)Gy → Fb' is false ('(∃y)Gy' is true because 'Ga' is true, 'Fb' is false). '(∀x)(∃y)(Gy → Fx)' is true because '(∃y)(Gy → Fa)' and '(∃y)(Gy → Fb)' are both ⊤, respectively because 'Gb → Fa' and 'Gb → Fb' are ⊤.

(II.2) Suppose $\mathcal{I}$ is an interpretation which verifies the sentence σ, that $\mathcal{I}$'s domain D contains n objects, and that $F_1 \dots F_k$ are all the monadic predicates to which $\mathcal{I}$ assigns an extension from D. Choose any element x in D. Then x has a *signature* in $\mathcal{I}$, which we can take to be a k-membered sequence of pluses and minuses, where a plus in j th position in the signature indicates that x is in the extension of $F_j$ and a minus that it is not in the extension of $F_j$. Two objects in D are said to be *indistinguishable* if they have the same signature (in terms of matrices, x and y have the same signature if they have the same pattern of pluses and minuses across their rows in the matrix). Though it can be proved rigorously, reflection on examples should be sufficient to convince the reader that adding an object to an interpretation and giving it the same signature as an object already in the interpretation does not affect the truth-value of any sentence in the interpretation. Hence, if $\mathcal{I}$ verifies σ and $\mathcal{I}$'s domain contains *n* objects, expanding $\mathcal{I}$ by adding an object with the same signature as one already in $\mathcal{I}$ yields an interpretation with *n* + 1 objects on which σ is still true.

## Chapter 6, §3

| | | | | |
|---|---|---|---|---|
| (I.3) | 1 | (1) | (∀x)(Fx → Gx) | Premise |
| | 2 | (2) | (∀x)Fx | Assumption |
| | 1 | (3) | Fa → Ga | 1 ∀E |
| | 2 | (4) | Fa | 2 ∀E |
| | 1,2 | (5) | Ga | 3,4 →E |
| | 1,2 | (6) | (∀x)Gx | 5 ∀I |
| | 1 | (7) | (∀x)Fx → (∀x)Gx | 2,6 →I ◆ |

| | | | | |
|---|---|---|---|---|
| (I.7) | 1 | (1) | (∃x)Fx → Ga | Premise |
| | 2 | (2) | Fa | Assumption |
| | 2 | (3) | (∃x)Fx | 2 ∃I |
| | 1,2 | (4) | Ga | 1,3 →E |
| | 1 | (5) | Fa → Ga | 2,4 →I |
| | 1 | (6) | (∃x)(Fx → Gx) | 5 ∃I ◆ |

(II.2) All tigers are fierce: $(\forall x)(Tx \rightarrow Fx)$
No antelope is fierce: $(\forall x)(Ax \rightarrow \sim Fx)$
No antelope is a tiger: $(\forall x)(Ax \rightarrow \sim Tx)$

This symbolization is the most convenient. If you symbolize 'no antelope is fierce' as '$\sim(\exists x)(Ax \;\&\; Fx)$' you will probably have trouble finding a proof.

| | | | |
|---|---|---|---|
| 1 | (1) | $(\forall x)(Tx \rightarrow Fx)$ | Premise |
| 2 | (2) | $(\forall x)(Ax \rightarrow \sim Fx)$ | Premise |
| 3 | (3) | $Ac$ | Assumption |
| 2 | (4) | $Ac \rightarrow \sim Fc$ | 2 ∀E |
| 2,3 | (5) | $\sim Fc$ | 4,3 →E |
| 1 | (6) | $Tc \rightarrow Fc$ | 1 ∀E |
| 1,2,3 | (7) | $\sim Tc$ | 6,5 SI (MT) |
| 1,2 | (8) | $Ac \rightarrow \sim Tc$ | 3,7 →I |
| 1,2 | (9) | $(\forall x)(Ax \rightarrow \sim Tx)$ | 8 ∀I ◆ |

## Chapter 6, §4

(I.2)

| | | | |
|---|---|---|---|
| 1 | (1) | $(\exists x)Fx \vee (\exists x)Gx$ | Premise |
| 2 | (2) | $(\exists x)Fx$ | Assumption |
| 3 | (3) | $Fa$ | Assumption |
| 3 | (4) | $Fa \vee Ga$ | 3 ∨I |
| 3 | (5) | $(\exists x)(Fx \vee Gx)$ | 4 ∃I |
| 2 | (6) | $(\exists x)(Fx \vee Gx)$ | 2,3,5 ∃I |
| 7 | (7) | $(\exists x)Gx$ | Assumption |
| 8 | (8) | $Ga$ | Assumption |
| 8 | (9) | $Fa \vee Ga$ | 8 ∨I |
| 8 | (10) | $(\exists x)(Fx \vee Gx)$ | 9 ∃I |
| 7 | (11) | $(\exists x)(Fx \vee Gx)$ | 7,8,10 ∃E |
| 1 | (12) | $(\exists x)(Fx \vee Gx)$ | 1,2,6,7,11 ∨E ◆ |

(I.8)

| | | | |
|---|---|---|---|
| 1 | (1) | $(\exists x)(Fx \vee (Gx \;\&\; Hx))$ | Premise |
| 2 | (2) | $(\forall x)(\sim Gx \vee \sim Hx)$ | Premise |
| 3 | (3) | $Fa \vee (Ga \;\&\; Ha)$ | Assumption |
| 2 | (4) | $\sim Ga \vee \sim Ha$ | 2 ∀E |
| 2 | (5) | $\sim(Ga \;\&\; Ha)$ | 4 SI (DeM) |
| 2,3 | (6) | $Fa$ | 3,5 SI (DS) |
| 2,3 | (7) | $(\exists x)Fx$ | 6 ∃I |
| 1,2 | (8) | $(\exists x)Fx$ | 1,3,7 ∃E ◆ |

(I.13)

| | | | |
|---|---|---|---|
| 1 | (1) | $(\exists x)Gx$ | Premise |
| 2 | (2) | $Ga$ | Assumption |
| 2 | (3) | $Fb \rightarrow Ga$ | 2 SI (PMI) |
| 2 | (4) | $(\exists y)(Fb \rightarrow Gy)$ | 3 ∃I |
| 2 | (5) | $(\forall x)(\exists y)(Fx \rightarrow Gy)$ | 4 ∀I |
| 1 | (6) | $(\forall x)(\exists y)(Fx \rightarrow Gy)$ | 1,2,5 ∃E ◆ |

| | | | | |
|---|---|---|---|---|
| (I.17) | 1 | (1) | $\sim(\exists x)(\forall y)(Fx \rightarrow Fy)$ | Assumption |
| | 2 | (2) | $\sim Fa$ | Assumption |
| | 2 | (3) | $Fa \rightarrow Fb$ | 2 SI (PMI) |
| | 2 | (4) | $(\forall y)(Fa \rightarrow Fy)$ | 3 $\forall$I |
| | 2 | (5) | $(\exists x)(\forall y)(Fx \rightarrow Fy)$ | 4 $\exists$I |
| | 1,2 | (6) | $\curlywedge$ | 1,5 ~E |
| | 1 | (7) | $\sim\sim Fa$ | 2, 6 ~I |
| | 1 | (8) | $Fa$ | 7 DN |
| | 1 | (9) | $Fb \rightarrow Fa$ | 8 SI (PMI) |
| | 1 | (10) | $(\forall y)(Fb \rightarrow Fy)$ | 9 $\forall$I |
| | 1 | (11) | $(\exists x)(\forall y)(Fx \rightarrow Fy)$ | 12 $\exists$I |
| | 1 | (12) | $\curlywedge$ | 1,11 ~E |
| | | (13) | $\sim\sim(\exists x)(\forall y)(Fx \rightarrow Fy)$ | 1,12 ~I |
| | | (14) | $(\exists x)(\forall y)(Fx \rightarrow Fy)$ | 13 DN ◆ |

| | | | | |
|---|---|---|---|---|
| (II.2 (↦)) | 1 | (1) | $(\exists x)(A \vee Fx)$ | Premise |
| | 2 | (2) | $A \vee Fb$ | Assumption |
| | 3 | (3) | $A$ | Assumption |
| | 3 | (4) | $A \vee (\exists x)Fx$ | 3 $\vee$I |
| | 5 | (5) | $Fb$ | Assumption |
| | 5 | (6) | $(\exists x)Fx$ | 5 $\exists$I |
| | 5 | (7) | $A \vee (\exists x)Fx$ | 6 $\vee$I |
| | 2 | (8) | $A \vee (\exists x)Fx$ | 2,3,4,5,7 $\vee$E |
| | 1 | (9) | $A \vee (\exists x)Fx$ | 1,2,8 $\exists$E ◆ |

| | | | | |
|---|---|---|---|---|
| (III.1) | | (1) | $(\exists y)Fy \vee \sim(\exists y)Fy$ | TI (LEM) |
| | 2 | (2) | $(\exists y)Fy$ | Assumption |
| | 3 | (3) | $Fb$ | Assumption |
| | 3 | (4) | $(\exists y)Fy \rightarrow Fb$ | 3 SI (PMI) |
| | 3 | (5) | $(\exists x)[(\exists y)Fy \rightarrow Fx]$ | 4 $\exists$I |
| | 2 | (6) | $(\exists x)[(\exists y)Fy \rightarrow Fx]$ | 2,3,5 $\exists$E |
| | 7 | (7) | $\sim(\exists y)Fy$ | Assumption |
| | 7 | (8) | $(\exists y)Fy \rightarrow Fb$ | 7 SI (PMI) |
| | 7 | (9) | $(\exists x)[(\exists y)Fy \rightarrow Fx]$ | 8 $\exists$I |
| | | (10) | $(\exists x)[(\exists y)Fy \rightarrow Fx]$ | 1,2,6,7,9 $\vee$E ◆ |

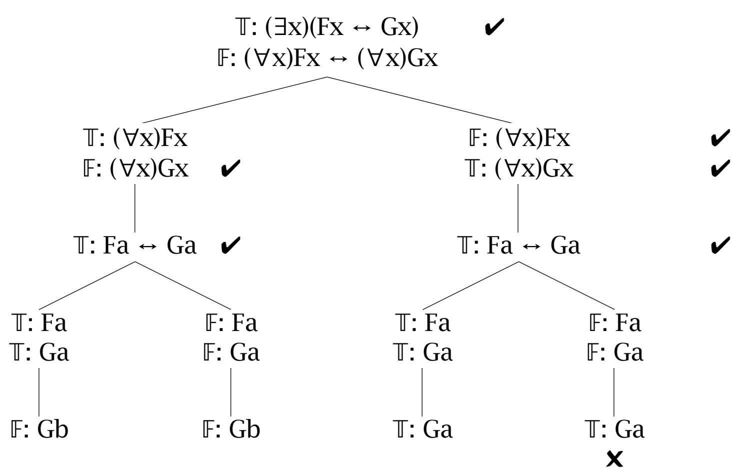
## Chapter 6, §5

| | | | | |
|---|---|---|---|---|
| (8) | 1 | (1) | $\sim(\exists x)(\forall y)(Fx \rightarrow Fy)$ | Assumption |
| | 1 | (2) | $(\forall x)\sim(\forall y)(Fx \rightarrow Fy)$ | 1 SI (QS) |
| | 1 | (3) | $\sim(\forall y)(Fa \rightarrow Fy)$ | 2 $\forall$E |
| | 1 | (4) | $(\exists y)\sim(Fa \rightarrow Fy)$ | 3 SI (QS) |
| | 5 | (5) | $\sim(Fa \rightarrow Fb)$ | Assumption |
| | 5 | (6) | $Fa \;\&\; \sim Fb$ | 5 SI (Neg-Imp) |
| | 5 | (7) | $\sim Fb$ | 6 &E |
| | 5 | (8) | $Fb \rightarrow Fc$ | 7 SI (PMI) |
| | 5 | (9) | $(\forall y)(Fb \rightarrow Fy)$ | 8 $\forall$I |

| | | | |
|---|---|---|---|
| 5 | (10) | $(\exists x)(\forall y)(Fx \rightarrow Fy)$ | 9 ∃I |
| 1,5 | (11) | $\curlywedge$ | 1,10 ~E |
| 1 | (12) | $\curlywedge$ | 4,5,11 ∃E |
| | (13) | $\sim\sim(\exists x)(\forall y)(Fx \rightarrow Fy)$ | 1,12 ~I |
| | (14) | $(\exists x)(\forall y)(Fx \rightarrow Fy)$ | 13 DN ◆ |

## Chapter 6, §7

(19) Determine whether $(\exists x)(Fx \leftrightarrow Gx) \vDash (\forall x)Fx \leftrightarrow (\forall x)Gx$.




Since three branches do not close, $(\exists x)(Fx \leftrightarrow Gx) \nvDash (\forall x)Fx \leftrightarrow (\forall x)Gx$.

## Chapter 6, §8

(I.2) Show $(\exists x)Fx \lor (\exists x)Gx \vdash_{NK} (\exists x)(Fx \lor Gx)$.

$$\dfrac{(\exists x)Fx \lor (\exists x)Gx \qquad \dfrac{\overset{(3)}{(\exists x)Fx} \quad \dfrac{\dfrac{\overset{(1)}{Fa}}{Fa \lor Ga}\,{\scriptstyle \lor I}}{(\exists x)(Fx \lor Gx)}\,{\scriptstyle \exists I}}{(\exists x)(Fx \lor Gx)}\,{\scriptstyle \exists E\,(1)} \qquad \dfrac{\overset{(4)}{(\exists x)Gx} \quad \dfrac{\dfrac{\overset{(2)}{Ga}}{Fa \lor Ga}\,{\scriptstyle \lor I}}{(\exists x)(Fx \lor Gx)}\,{\scriptstyle \exists I}}{(\exists x)(Fx \lor Gx)}\,{\scriptstyle \exists E\,(2)}}{(\exists x)(Fx \lor Gx)}\,{\scriptstyle \lor E\,(3,4)}$$

## Chapter 7, §1

(I.3) '~(∃x) no one loves x'; 'no one loves x' = '~(∃y)Lyx'; so '~(∃x)~(∃y)Lyx' or '(∀x)(∃y)Lyx'.

(I.6) '(∀x)(if x loves x then x is a lover)'. 'x is a lover' = 'x loves someone' = '(∃y)Lxy'. So: '(∀x)(Lxx → (∃y)Lxy)'.

(III.4) '(∃x) x is a student and x does not read any books'; 'x does not read any books' = '(∀y) if y is a book then x does not read y'. So: '(∃x)(Sx & (∀y)(By → ~Rxy))'. Domain: things.

(III.12) '(∀x) if x is a play attributed to Shakespeare then Marlowe wrote x'; 'x is a play attributed to Shakespeare' = 'x is a play and x is attributed to Shakespeare'. So: '(∀x)((Px & Axa) → Wbx)'. Domain: things.

(III.17) '(∃x) x is a number and (x is prime if and only if all numbers ⩾ x are composite)'. 'all numbers ⩾ x are composite' = '(∀y) if y is at least as large as x then y is not prime'. So: '(∃x)(Nx & [Px ↔ (∀y)((Ny & Qyx) → ~Px)])'. [N.B.: The statement is true—let x be any composite number. Then because there are infinitely many primes, both sides of the biconditional are false.] Domain: things.

(III.22) '~(∃x) x is a person who trusts a politician who makes promises he can't keep' = '~(∃x)(Px & (∃y)(y is a politician who makes promises y can't keep & x trusts y))'. 'y is a politician who makes promises y can't keep' = 'Ly & (∃z) z is a promise & y makes z & y can't keep z'. So: '~(∃x)(∃y)(Px & (Ly & [(∃z)(Rz & (Myz & ~Kyz)) & Txy]))'. Domain: things.

(III.27) '(∃x)(x is a composer and x is liked by anyone who likes any composer at all)'; 'x is liked by anyone who likes any composer at all' = '(∀y)(if y likes any composer at all then y likes x)'; 'y likes any composer at all' = 'y likes at least one composer'. So: '(∃x)(Cx & (∀y)[(∃z)(Cz & Lyz) → Lyx])'. Domain: people.

(III.30) '(∀x) if x pities those who pity themselves then x is lonely'. 'x pities those who pity themselves' = '(∀y)(y pities y → x pities y)'. So: '(∀x)[(∀y)(Pyy → Pxy) → Lx]'. Domain: people.

(IV.2) 'Some sequents have only finite counterexamples.'

## Chapter 7, §2

(I.6) '~(∃x) x is wiser than someone else'; 'x is wiser than someone else' = '(∃y)(y ≠ x & Wxy)'. So: '~(∃x)(∃y)(y ≠ x & Wxy)'. (The English means that all people are equally wise.)

(I.13) '(∀x)(∀y)(∀z)(if (Cx & Cy & Cz) then (if x answers every question & y answers every question & z answers every question then x = y ∨ y = z ∨ x = z))'; 'x answers every question' = '(∀w)(Qw → Axw)', 'y answers every question' = '(∀w)(Qw → Ayw)', 'z answers every question' = '(∀w)(Qw → Azw)'. So:

$$(\forall x)(\forall y)(\forall x)\{(Cx\ \&\ Cy\ \&\ Cz) \to ([(\forall w)(Qw \to Axw)\ \&\ (\forall w)(Qw \to Ayw)\ \& \\ (\forall w)(Qw \to Azw)] \to (x = y \vee y = z \vee x = z))\}.$$

We can use a single quantifier for the three "every"'s, which allows us to simplify the antecedent of the internal conditional to '(∀w)(Qw → (Axw & Ayw & Azw))'.

(18) There is exactly one composer liked by anyone who likes any composer at all and he is (identical to) Mozart.

'(∃x) (x is a composer liked by anyone who likes any composer at all and (∀y)(if y is a composer liked by anyone who likes any composer at all then y = x) and x = Mozart))'.

'x is a composer liked by anyone who likes any composer at all' = '(∀y)(if y likes any composer at all then y likes x)'; 'y likes any composer at all' = 'y likes at least one composer' = '(∃z)(Cz & Lyz)'. For this treatment of 'any', see the discussion of Examples 5.3.18–5.3.20 on page 162.

So: 'x is liked by anyone who likes any composer at all' = '(∀y)[(∃z)(Cz & Lyz) → Lyx]'. Thus 'y is a composer liked by anyone who likes any composer at all' = '(∀w)[(∃z)(Cz & Lwz) → Lwy]'. So for the whole formula:

$$(\exists x)[\{Cx\ \&\ (\forall y)([(\exists z)(Cz\ \&\ Lyz) \to Lyx]\}\ \& \\ (\forall y)(\{Cy\ \&\ (\forall w)[(\exists z)(Cz\ \&\ Lwz) \to Lwy]\} \to y = x)\ \&\ x = m].$$

## Chapter 7, §3

(I.3) The left parenthesis between the two quantifiers should not be there if the subformula following it was formed by an application of (*f-q*), since (*f-q*) does not put outer parentheses around the formulae formed using it. And either this parenthesis or the one immediately preceding 'Rxy' has no matching right parenthesis.

## Chapter 7, §4

(I.4) The two readings are (a) that he would sell to no one, and (b) that he would not sell to just anyone, that is, there are certain people he would not sell to.

(a) John would sell a picture to no one: (∀x)(Tx → ~(∃y)(Py & Sjxy))
(b) John would not sell to just anyone: (∀x)(Tx → (∃y)~(Py & Sjxy)).

So the ambiguity lies in the relative scopes of '∃' and '~'.

(II.2) The three readings are: (a) only *private universities* are expensive (nothing else is expensive); (b) only *private* universities are expensive (no other kind of university is); (c) only private *universities* are expensive (no other private things are).

(a) (∀x)(Ex → (Px & Ux)); (b) (∀x)((Ex & Ux) → Px); (c) (∀x)((Ex & Px) → Ux).

## Chapter 8, §1

(I.5) '(∀x)(Rxx → (∃z)Sxz)' is false because 'Rdd → (∃z)Sdz' is false, since 'Rdd' is true and '(∃z)Sdz' is false. '(∃z)Sdz' is false because 'Sda', 'Sdb', 'Sdc' and 'Sdd' are all false, since none of ⟨δ,α⟩, ⟨δ,β⟩, ⟨δ,γ⟩ or ⟨δ,δ⟩ belongs to Ext(S).

(II.4) '(∀x)(∀y)(Rxy ↔ ~Syx)' is false because '(∀y)(Ray ↔ ~Sya)' is false, because 'Rab ↔ ~Sba' is false, because 'Rab' is true and 'Sba' is true.

(III.4) '(∃x)(∀y)(y ≠ a → (∃z)Sxyz)' is true because '(∀y)(y ≠ a → (∃z)S1yz)' is true, since for every element y from the domain other than 10, z = 1 + y is a member of the domain.

(IV.3) '(∀x)(x ≠ 1 → (∃y) y > x)' is true since for any number n ∈ [0,1], '**n** ≠ 1 → (∃y) y > **n**' is true; when n = 1 the antecedent is false, and for every other *n*, '(∃y) y > **n**' is true since '1 > **n**' is true. Here we use boldface 'n' to mean the standard numeral for the number *n*. (In fact, for any particular n ≠ 1, there are infinitely many numbers y in [0,1] such that y > n, but 1 can be used in every case).

## Chapter 8, §2

(I.4) D = {α,β}, Ext(R) = {⟨α,β⟩, ⟨β,β⟩}. '(∃x)~Rxb' is false because '~Rab' is false and '~Rbb' is false. '(∀x)(Rxa → ~Rxb)' is true because 'Raa → ~Rab' and 'Rba → ~Rbb' are both ⊤. '(∀x)(∃y)Rxy' is true because '(∃y)Ray' and '(∃y)Rby' are both ⊤, respectively because 'Rab' and 'Rbb' are both ⊤.

(I.12) D = {α,β}, Ext(F) = Ext(G) = Ext(H) = {α}, Ext(R) = {⟨α,α⟩}. '(∀x)(∀y)((Fx & Hx) → Rxy)' is false because '(Fa & Ha) → Rab' is false. '(∀x)(Hx → Gx)' is true because 'Ha → Ga' and 'Hb → Gb' are true. '(∀x)(Fx → (∀y)(Gy → Rxy))' is true because 'Fb → (∀y)(Gy → Rby)' is true (false antecedent) and 'Fa → (∀y)(Gy → Ray)' is true because 'Fa' is true and '(∀y)(Gy → Ray)' is true. '(∀y)(Gy → Ray)' is true since 'Gb → Rab' is true (false antecedent) and 'Ga → Raa' is true (⊤ → ⊤).

(II.3) D = {α}, Ext(F) = {α}. Then 'Ga' is false, and 'Ga → Fa' is true so '(∀x)(Gx → Fx)' is true. α is the only thing which is F, so the first premise is also true.

(III.2) D = {α,β}, Ext(∈) = {⟨α,β⟩, ⟨β,α⟩}, Ext(S) = {α,β}. (E) is true since α ≠ β, α ∈ β, β ∉ β. But ‘(∃x)(∀y)y ∉ x’ is false since ‘(∀y)y ∉ a’ and ‘(∀y)y ∉ b’ are both false.

## Chapter 8, §3

| | | | | |
|---|---|---|---|---|
| (I.3) | 1 | (1) | (∃x)Fx → (∀x)(Fx → Gxa) | Premise |
| | 2 | (2) | (∃x)Hx → (∀x)(Hx → Jxa) | Premise |
| | 3 | (3) | (∃x)(Fx & Hx) | Assumption |
| | 4 | (4) | Fa & Ha | Assumption |
| | 4 | (5) | Fa | 4 &E |
| | 4 | (6) | Ha | 4 &E |
| | 4 | (7) | (∃x)Fx | 5 ∃I |
| | 1,4 | (8) | (∀x)(Fx → Gxa) | 1,7 →E |
| | 4 | (9) | (∃x)Hx | 6 ∃I |
| | 2,4 | (10) | (∀x)(Hx → Jxa) | 2,9 →E |
| | 1,4 | (11) | Fa → Gaa | 8 ∀E |
| | 1,4 | (12) | Gaa | 11,5 →E |
| | 2,4 | (13) | Ha → Jaa | 10 ∀E |
| | 2,4 | (14) | Jaa | 13,6 →E |
| | 1,2,4 | (15) | Gaa & Jaa | 12,14 &I |
| | 1,2,4 | (16) | (∃y)(Gay & Jay) | 15 ∃I |
| | 1,2,4 | (17) | (∃x)(∃y)(Gxy & Jxy) | 16 ∃I |
| | 1,2,3 | (18) | (∃x)(∃y)(Gxy & Jxy) | 3,4,17 ∃E |
| | 1,2 | (19) | (∃x)(Fx & Hx) → (∃x)(∃y)(Gxy & Jxy) | 3,18 →I ◆ |

| | | | | |
|---|---|---|---|---|
| (I.7) | 1 | (1) | (∀x)(∀y)(Exy → Eyx) | Premise |
| | 2 | (2) | (∀x)(∀y)(Exy → Exx) | Premise |
| | 3 | (3) | (∃y)Eya | Assumption |
| | 4 | (4) | Eba | Assumption |
| | 1 | (5) | (∀y)(Eby → Eyb) | 1 ∀E |
| | 1 | (6) | Eba → Eab | 5 ∀E |
| | 1,4 | (7) | Eab | 6,4 →E |
| | 2 | (8) | (∀y)(Eay → Eaa) | 2 ∀E |
| | 2 | (9) | Eab → Eaa | 8 ∀E |
| | 1,2,4 | (10) | Eaa | 9,7 →E |
| | 1,2,3 | (11) | Eaa | 3,4,10 ∃E |
| | 1,2 | (12) | (∃y)Eya → Eaa | 3,11 →I |
| | 1,2 | (13) | (∀x)[(∃y)Eyx → Exx] | 12 ∀I ◆ |

Note that ∃E at (11) is legal since the name used to form the instance of (3) is ‘b’, and ‘b’ does not occur in 3, 10, or 1 and 2.

| | | | | |
|---|---|---|---|---|
| (I.12) | 1 | (1) | (∃x)(∀y)((Fx ∨ Gy) → (∀z)(Hxy → Hyz)) | Premise |
| | 2 | (2) | (∃z)(∀x)~Hxz | Premise |
| | 3 | (3) | (∀y)((Fa ∨ Gy) → (∀z)(Hay → Hyz)) | Assumption |
| | 4 | (4) | (∀x)~Hxb | Assumption |

| | | | |
|---|---|---|---|
| 5 | (5) | Fa | Assumption |
| 3 | (6) | (Fa ∨ Gc) → (∀z)(Hac → Hcz) | 3 ∀E |
| 5 | (7) | Fa ∨ Gc | 5 ∨I |
| 3,5 | (8) | (∀z)(Hac → Hcz) | 6,7 →E |
| 3,5 | (9) | Hac → Hcb | 8 ∀E |
| 4 | (10) | ~Hcb | 4 ∀E |
| 3,4,5 | (11) | ~Hac | 9,10 SI (MT) |
| 3,4 | (12) | Fa → ~Hac | 5,11 →I |
| 3,4 | (13) | (∀x)(Fa → ~Hax) | 12 ∀I |
| 3,4 | (14) | (∃y)(∀x)(Fy → ~Hyx) | 13 ∃I |
| 1,3 | (15) | (∃y)(∀x)(Fy → ~Hyx) | 2,4,14 ∃E |
| 1,2 | (16) | (∃y)(∀x)(Fy → ~Hyx) | 1,3,15 ∃E ◆ |

(II.4) T_: _ is tall; A_: _ applied; L_,_: _ is taller than _; b: John; c: Mary

| | | | |
|---|---|---|---|
| 1 | (1) | (∀x)(Ax → ~Tx) | Premise |
| 2 | (2) | Tc & Lbc | Premise |
| 3 | (3) | (∀x)[(∃y)(Ty & Lxy) → Tx] | Premise |
| 4 | (4) | Ab | Assumption |
| 1 | (5) | Ab → ~Tb | 1 ∀E |
| 1,4 | (6) | ~Tb | 4,5 →E |
| 3 | (7) | (∃y)(Ty & Lby) → Tb | 3 ∀E |
| 2 | (8) | (∃y)(Ty & Lby) | 2 ∃I |
| 2,3 | (9) | Tb | 7,8 →E |
| 1,2,3,4 | (10) | ⅄ | 6,9 ~E |
| 1,2,3 | (11) | ~Ab | 4,10 ~I ◆ |

(II.7) S_: _ is a student; C_: _ cheated; P_: _ is a professor; B_,_: _ bribed _; A_: _ was accused

| | | | |
|---|---|---|---|
| 1 | (1) | (∀x)((Sx & Cx) → (∃y)(Py & Bxy)) | Premise |
| 2 | (2) | (∃x)[(Sx & Ax) & (∀y)((Py & Bxy) → Ay)] | Premise |
| 3 | (3) | (∀x)((Sx & Ax) → Cx) | Premise |
| 4 | (4) | (Sd & Ad) & (∀y)((Py & Bdy) → Ay) | Assumption |
| 4 | (5) | Sd & Ad | 4 &E |
| 3 | (6) | (Sd & Ad) → Cd | 3 ∀E |
| 3,4 | (7) | Cd | 6,5 →E |
| 4 | (8) | Sd | 5 &E |
| 3,4 | (9) | Sd & Cd | 8,7 &I |
| 1 | (10) | (Sd & Cd) → (∃y)(Py & Bdy) | 1 ∀E |
| 1,3,4 | (11) | (∃y)(Py & Bdy) | 10,9 →E |
| 12 | (12) | Pe & Bde | Assumption |
| 4 | (13) | (∀y)((Py & Bdy) → Ay) | 4 &E |
| 4 | (14) | (Pe & Bde) → Ae | 13 ∀E |
| 4,12 | (15) | Ae | 14,12 →E |
| 3 | (16) | (Se & Ae) → Ce | 3 ∀E |
| 17 | (17) | Se | Assumption |

| | | | |
|---|---|---|---|
| 4,12,17 | (18) | Se & Ae | 17,15 &I |
| 3,4,12,17 | (19) | Ce | 16,18 →E |
| 3,4,12 | (20) | Se → Ce | 17,19 →I |
| 3,4,12 | (21) | ~Se ∨ Ce | 20 SI (Imp) |
| 12 | (22) | Pe | 12 &E |
| 3,4,12 | (23) | Pe & (~Se ∨ Ce) | 22,21 &I |
| 3,4,12 | (24) | (∃x)(Px & (~Sx ∨ Cx)) | 23 ∃I |
| 1,3,4 | (25) | (∃x)(Px & (~Sx ∨ Cx)) | 11,12,24 ∃E |
| 1,2,3 | (26) | (∃x)(Px & (~Sx ∨ Cx)) | 2,4,25 ∃E ◆ |

## Chapter 8, §4

| | | | | |
|---|---|---|---|---|
| (I.7) | 1 | (1) | (∃x)(∃y)Hxy | Premise |
| | 2 | (2) | (∀y)(∀z)(Dyz ↔ Hzy) | Premise |
| | 3 | (3) | (∀x)(∀y)(~Hxy ∨ x = y) | Premise |
| | 4 | (4) | (∃y)Hay | Assumption |
| | 5 | (5) | Hab | Assumption |
| | 2 | (6) | Dba ↔ Hab | 2 ∀E twice |
| | 2,5 | (7) | Dba | 5,6 SI |
| | 3 | (8) | ~Hab ∨ a = b | 3 ∀E twice |
| | 3 | (9) | Hab → a = b | 8 SI (Imp) |
| | 3,5 | (10) | a = b | 9,5 →E |
| | 3,5 | (11) | Hbb | 10,5 =E |
| | 2,5 | (12) | Dbb | 10,7 =E |
| | 2,3,5 | (13) | Hbb & Dbb | 11,12 &I |
| | 2,3,5 | (14) | (∃x)(Hxx & Dxx) | 13 ∃I |
| | 2,3,4 | (15) | (∃x)(Hxx & Dxx) | 4,5,14 ∃E |
| | 1,2,3 | (16) | (∃x)(Hxx & Dxx) | 1,4,15 ∃E ◆ |

| | | | | |
|---|---|---|---|---|
| (III.2) | 1 | (1) | (∀x)(∀y)((Sx & Sy) → (x = y ↔ (∀z)(z ∈ x ↔ z ∈ y))) | Premise |
| | 2 | (2) | (Sa & Sb) & [~(∃z)z ∈ a & ~(∃z)z ∈ b] | Assumption |
| | 1 | (3) | (Sa & Sb) → (a = b ↔ (∀z)(z ∈ a ↔ z ∈ b)) | 1 ∀E |
| | 2 | (4) | Sa & Sb | 2 &E |
| | 1,2 | (5) | a = b ↔ (∀z)(z ∈ a ↔ z ∈ b) | 3,4 →E |
| | 2 | (6) | ~(∃z)z ∈ a & ~(∃z)z ∈ b | 2 &E |
| | 2 | (7) | ~(∃z)z ∈ a | 6 &E |
| | 2 | (8) | ~(∃z)z ∈ b | 6 &E |
| | 2 | (9) | (∀z)z ∉ a | 7 SI (QS) |
| | 2 | (10) | (∀z)z ∉ b | 8 SI (QS) |
| | 2 | (11) | c ∉ a | 9 ∀E |
| | 2 | (12) | c ∉ b | 10 ∀E |
| | 2 | (13) | c ∉ a & c ∉ b | 11,12 &I |
| | 2 | (14) | (c ∈ a & c ∈ b) ∨ (c ∉ a & c ∉ b) | 13 ∨I |
| | 2 | (15) | c ∈ a ↔ c ∈ b | 14 SI ((16), page 119) |

| | | | |
|---|---|---|---|
| 2 | (16) | $(\forall z)(z \in a \leftrightarrow z \in b)$ | 15 ∀I |
| 1,2 | (17) | $a = b$ | 5,16 SI |
| 1 | (18) | $((Sa \,\&\, Sb) \,\&\, [\sim(\exists z)z \in a \,\&\, \sim(\exists z)z \in b]) \rightarrow a = b$ | 2,17 →I |
| 1 | (19) | $(\forall y)[((Sa \,\&\, Sy) \,\&\, [\sim(\exists z)z \in a \,\&\, \sim(\exists z)z \in y]) \rightarrow a = y]$ | 18 ∀I |
| 1 | (20) | $(\forall x)(\forall y)[((Sx \,\&\, Sy) \,\&\, [\sim(\exists z)z \in x \,\&\, \sim(\exists z)z \in y]) \rightarrow x = y]$ | 19 ∀I ◆ |

## Chapter 8, §5

(I.4) 'is similar in color to' is reflexive and symmetric, but not transitive. Consider a sequence of objects each of which is similar in color to objects adjacent to it. The last object in the sequence may not be similar in color to the first. (See further Chapter Eleven.)

(I.6) 'semantically entails' is single-premise reflexive and transitive, since for every p, $p \vDash p$ and for every p, q, r, $p \vDash q$ and $q \vDash r$ imply that $p \vDash r$ (suppose that $p \vDash q$ and $q \vDash r$; then if some assignment makes p true and r false, that assignment makes q false, since $q \vDash r$, but then $p \nvDash q$, contrary to hypothesis). But 'semantically entails' is not symmetric: $A \,\&\, B \vDash A$ but $A \nvDash A \,\&\, B$.

(II.2) $D = \{\alpha,\beta\}$; $Ext(R) = \{\langle\alpha,\beta\rangle,\langle\beta,\alpha\rangle,\langle\alpha,\alpha\rangle,\langle\beta,\beta\rangle\}$. Then R is reflexive and transitive (note that '$(Rab \,\&\, Rba) \rightarrow Raa$' and '$(Rba \,\&\, Rab) \rightarrow Rbb$' are both true, and also totally connected, but R is not anti-symmetric: '$((Rab \,\&\, Rba) \rightarrow a = b)$' is false.

(II.6) $D = \{\alpha,\beta,\gamma\}$; $Ext(R) = \{\langle\alpha,\alpha\rangle,\langle\beta,\beta\rangle,\langle\gamma,\gamma\rangle,\langle\alpha,\beta\rangle,\langle a,\gamma\rangle\}$. Then R is reflexive and transitive, but not directed, since '$(Rab \,\&\, Rac) \rightarrow (\exists w)(Rbw \,\&\, Rcw)$' is false. '$Rab \,\&\, Rac$' is true but '$(\exists w)(Rbw \,\&\, Rcw)$' is false, since '$Rba \,\&\, Rca$', '$Rbb \,\&\, Rcb$' and '$Rbc \,\&\, Rcc$' are all false.

| | | | | |
|---|---|---|---|---|
| (III.2) | 1 | (1) | $(\forall x)(\forall y)(\forall z)((Rxy \,\&\, Ryz) \rightarrow Rxz)$ | Premise |
| | 2 | (2) | $(\forall x)(\forall y)(\forall z)((Rxy \,\&\, Ryz) \rightarrow \sim Rxz)$ | Premise |
| | 3 | (3) | $Rab$ | Assumption |
| | 4 | (4) | $Rba$ | Assumption |
| | 3,4 | (5) | $Rab \,\&\, Rba$ | 3,4 &I |
| | 1 | (6) | $(Rab \,\&\, Rba) \rightarrow Raa$ | 1 ∀E thrice |
| | 1,3,4 | (7) | $Raa$ | 6,5 →E |
| | 2 | (8) | $(Rab \,\&\, Rba) \rightarrow \sim Raa$ | 2 ∀E thrice |
| | 2,3,4 | (9) | $\sim Raa$ | 8,5 →E |
| | 1,2,3,4 | (10) | $\curlywedge$ | 9,7 ~E |
| | 1,2,3 | (11) | $\sim Rba$ | 4,10 ~I |
| | 1,2 | (12) | $Rab \rightarrow \sim Rba$ | 3,11 →I |
| | 1,2 | (13) | $(\forall y)(Ray \rightarrow Rya)$ | 12 ∀I |
| | 1,2 | (14) | $(\forall x)(\forall y)(Rxy \rightarrow \sim Ryx)$ | 13 ∀I ◆ |

(IV)

| | | | | |
|---|---|---|---|---|
| | 1 | (1) | $(\forall x)(\forall y)(\forall z)((Rxy \& Rxz) \rightarrow Ryz)$ | Premise |
| | 2 | (2) | $Rab$ | Assumption |
| | 1 | (3) | $(Rab \& Rab) \rightarrow Rbb$ | 1 $\forall$E thrice |
| | 2 | (4) | $Rab \& Rab$ | 2,2 &I |
| | 1,2 | (5) | $Rbb$ | 3,4 $\rightarrow$E |
| | 1 | (6) | $Rab \rightarrow Rbb$ | 2,5 $\rightarrow$I |
| | 1 | (7) | $(\forall y)(Ray \rightarrow Ryy)$ | 6 $\forall$I |
| | 1 | (8) | $(\forall x)(\forall y)(Rxy \rightarrow Ryy)$ | 7 $\forall$I ◆ |

## Chapter 8, §8

(I.2) $(\exists X)(Xa \& \sim Xb)$. (I.5) $(\forall x)(Xa \leftrightarrow Xb)$.

(II.2) Let D = {1,2,3,…,10}, Ext(F) = {1,2,3,4,5,6}, Ext(G) = {1,2}. Then ‘Few Fs are G’ is true because only two of the six Fs are G. But ‘$(\mathcal{W}x)(Fx \rightarrow Gx)$’ is false since it has six true instances on this interpretation. Therefore ‘$(\mathcal{W}x)(Fx \rightarrow Gx)$’ is not a correct symbolization of ‘Few Fs are G’.

## Chapter 9, §2

(6) $\mathcal{W} = (w^*, u)$; $w^*$: $A \mapsto \bot$, $B \mapsto \bot$; u: $A \mapsto \top$, $B \mapsto \bot$. Then $w^*[A \rightarrow B] = \top$, hence $w^*[\diamond(A \rightarrow B)] = \top$. But $u[A] = \top$ so $w^*[\diamond A] = \top$, while $w^*[B] = \bot$ and $u[B] = \bot$, so $w^*[\diamond B] = \bot$. Therefore $w^*[\diamond A \rightarrow \diamond B] = \bot$.

(11) $\mathcal{W} = (w^*, u)$; $w^*$: $A \mapsto \bot$, $B \mapsto \top$ (or $\bot$); u: $A \mapsto \top$, $B \mapsto \bot$. $u[A \rightarrow B] = \bot$, hence $w^*[\Box(A \rightarrow B) = \bot$. But $w^*[\Box A] = \bot$ since $w^*[A] = \bot$. Hence $w^*[\Box A \rightarrow \diamond B] = \top$.

## Chapter 9, §3

(I.4) $\text{Trans}[\Box(A \rightarrow \diamond\curlywedge)] = (\forall w)\text{Trans}[A \rightarrow \diamond\curlywedge] = (\forall w)(\text{Trans}[A] \rightarrow \text{Trans}[\diamond\curlywedge]) = (\forall w)(A'w \rightarrow (\exists w)\text{Trans}[\curlywedge]) = (\forall w)(A'w \rightarrow (\exists w)\curlywedge)$.

(II.2) ‘$(\exists w)(\forall w)A'w \rightarrow A'w$’ is the translation of the conditional ‘$\diamond\Box A \rightarrow A$’.

## Chapter 9, §4

(6)

| | | | | |
|---|---|---|---|---|
| | 1 | (1) | $\Box(A \& \diamond B)$ | Premise |
| | 1 | (2) | $A \& \diamond B$ | 1 $\Box$E |
| | 1 | (3) | $A$ | 2 &E |
| | 1 | (4) | $\diamond B$ | 2 &E |
| | 5 | (5) | $B$ | Assumption |
| | 1,5 | (6) | $A \& B$ | 3,5 &I |
| | 1,5 | (7) | $\diamond(A \& B)$ | 6 $\diamond$I |
| | 1 | (8) | $\diamond(A \& B)$ | 4,5,7 $\diamond$E ◆ |

(19)

| | | | |
|---|---|---|---|
| 1 | (1) | $\Box(A \to \Box B)$ | Premise |
| 2 | (2) | $\Diamond A$ | Assumption |
| 3 | (3) | $A$ | Assumption |
| 1 | (4) | $A \to \Box B$ | 1 $\Box$E |
| 1,3 | (5) | $\Box B$ | 4,3 $\to$E |
| 1,2 | (6) | $\Box B$ | 2,3,5 $\Diamond$E |
| 1,2 | (7) | $B$ | 6 $\Box$E |
| 1 | (8) | $\Diamond A \to B$ | 2,7 $\to$I |
| | (9) | $\Box(\Diamond A \to B)$ | 8 $\Box$I ◆ |

## Chapter 9, §5

(5) $W = \{w^*,u\}$, $w^*(D) = \{\alpha\}$, $u(D) = \{\alpha\}$ (or $\varnothing$), $w^*[F] = \{\alpha\}$, $w^*[G] = \varnothing$, $u[F] = \varnothing$, $u[G] = \{\alpha\}$. Then $w^*[\Box Fa] = \bot$, $w^*[\Box Ga] = \bot$, so $w^*[(\forall x)(\Box Fx \leftrightarrow \Box Gx)] = \top$. However, $w^*[Fa \leftrightarrow Ga] = \bot$ so $w^*[\Box(Fa \leftrightarrow Ga)] = \bot$ so $w^*[(\forall x)\Box(Fx \leftrightarrow Gx)] = \bot$.

(10) $W = \{w^*,u\}$, $w^*(D) = \{\alpha\}$, $u(D) = \{\beta\}$. $w^*[R] = \varnothing$, $u[R] = \langle\alpha,\beta\rangle$. Then $u[(\exists y)Ray] = \top$ so $w^*[\Diamond(\exists y)Ray] = \top$. $\alpha \in w^*(D)$, so $w^*[(\exists x)\Diamond(\exists y)Rxy] = \top$. However, $w^*[(\exists x)(\exists y)Rxy] = \bot$ and also $u[(\exists x)(\exists y)Rxy] = \bot$ (‘$(\exists y)Ray$’, though true at u, does not make ‘$(\exists x)(\exists y)Rxy$’ true at u since $\alpha \notin u(D)$). So $w^*[\Diamond(\exists x)(\exists y)Rxy] = \bot$.

(16) $W = \{w^*,u\}$, $w^*(D) = \varnothing$, $u(D) = \{\alpha\}$, $w^*[F] = \{\alpha\}$ (or $\varnothing$), $u[F] = \{\alpha\}$. Then $w^*[(\forall x)\Box Ex] = \top$ since $w^*(D) = \varnothing$, and $w^*[(\exists x)\Diamond Fx] = \bot$ for the same reason. But $u[(\exists x)Fx] = \top$ so $w^*[\Diamond(\exists x)Fx] = \top$.

## Chapter 9, §6

(5)

| | | | |
|---|---|---|---|
| 1 | (1) | $(\forall x)\Box Ex$ | Premise |
| 2 | (2) | $(\exists x)\Diamond Fx$ | Assumption |
| 3 | (3) | $Ea \,\&\, \Diamond Fa$ | Assumption |
| 3 | (4) | $Ea$ | 3 &E |
| 1 | (5) | $Ea \to \Box Ea$ | 1 $\forall$E |
| 1,3 | (6) | $\Box Ea$ | 5,4,$\to$E |
| 7 | (7) | $\Box Ea$ | Assumption |
| 3 | (8) | $\Diamond Fa$ | 3 &E |
| 9 | (9) | $Fa$ | Assumption |
| 7 | (10) | $Ea$ | 7 $\Box$E |
| 7,9 | (11) | $Ea \,\&\, Fa$ | 10, 9 &I |
| 7,9 | (12) | $(\exists x)Fx$ | 11 $\exists$I |
| 7,9 | (13) | $\Diamond(\exists x)Fx$ | 12 $\Diamond$I |
| 7,3 | (14) | $\Diamond(\exists x)Fx$ | 8,9,13 $\Diamond$E |
| 3 | (15) | $\Box Ea \to \Diamond(\exists x)Fx$ | 7,14 $\to$I |
| 1,3 | (16) | $\Diamond(\exists x)Fx$ | 15,6 $\to$E |
| 1,2 | (17) | $\Diamond(\exists x)Fx$ | 2,3,16 $\exists$E |
| 1 | (18) | $(\exists x)\Diamond Fx \to \Diamond(\exists x)Fx$ | 2,17 $\to$I ◆ |

| | | | | |
|---|---|---|---|---|
| (12) | 1 | (1) | ◇(∀x)□Ex | Premise |
| | 2 | (2) | □(∀x)Fx | Assumption |
| | 3 | (3) | (∀x)□Ex | Assumption |
| | 3 | (4) | Ea → □Ea | 3 ∀E |
| | 2 | (5) | (∀x)Fx | 2 □E |
| | 2 | (6) | Ea → Fa | 5 ∀E |
| | 7 | (7) | □Ea | Assumption |
| | 7 | (8) | Ea | 7 □E |
| | 2,7 | (9) | Fa | 6,8 →E |
| | 2,7 | (10) | □Fa | 9 □I |
| | 2,7 | (11) | Ea → □Fa | 10 SI (PMI) |
| | 2 | (12) | □Ea → (Ea → □Fa) | 7,11 →I |
| | 2,3 | (13) | Ea → (Ea → □Fa) | 4,12 SI (Chain) |
| | 2,3 | (14) | Ea → □Fa | 13 SI (NK) |
| | 2,3 | (15) | (∀x)□Fx | 14 ∀I |
| | 3 | (16) | □(∀x)Fx → (∀x)□Fx | 2,15 →I |
| | 3 | (17) | ◇[□(∀x)Fx → (∀x)□Fx] | 16 ◇I |
| | 1 | (18) | ◇[□(∀x)Fx → (∀x)□Fx] | 1,3,17 ◇E ◆ |

| | | | | |
|---|---|---|---|---|
| (19) | 1 | (1) | □(∀x)□Ex | Premise |
| | 2 | (2) | A | Premise |
| | 3 | (3) | ◇(∀x)(Fx → □(A → ~Fx)) | Premise |
| | 4 | (4) | ◇A | Assumption |
| | 1 | (5) | □Ea | 1 SI (Example 3) |
| | 6 | (6) | (∀x)(Fx → □(A → ~Fx)) | Assumption |
| | 7 | (7) | □(A → ~Fa) | Assumption |
| | 8 | (8) | A | Assumption |
| | 7 | (9) | A → ~Fa | 7 □E |
| | 7,8 | (10) | ~Fa | 9, 8 →E |
| | 7,8 | (11) | ◇~Fa | 10 ◇I |
| | 4,7 | (12) | ◇~Fa | 4,8,11 ◇E |
| | 4 | (13) | □(A → ~Fa) → ◇~Fa | 7,12 →I |
| | | (14) | Fa ∨ ~Fa | TI (LEM) |
| | 15 | (15) | Fa | Assumption |
| | 6 | (16) | Ea → (Fa → □(A → ~Fa)) | 6 ∀E |
| | 1 | (17) | Ea | 5 □E |
| | 1,6 | (18) | Fa → □(A → ~Fa) | 16,17 →E |
| | 1,4,6 | (19) | Fa → ◇~Fa | 18,13 SI (Chain) |
| | 1,4,6,15 | (20) | ◇~Fa | 19,15 →E |
| | 21 | (21) | ~Fa | Assumption |
| | 21 | (22) | ◇~Fa | 21 ◇I |
| | 1,4,6 | (23) | ◇~Fa | 14,15,20,21,22 ∨E |
| | 1,3,4 | (24) | ◇~Fa | 3,6,23 ◇E |
| | 1,3,4 | (25) | Ea & ◇~Fa | 17,24 &I |
| | 1,3,4 | (26) | (∃x)◇~Fx | 25 ∃I |
| | 1,3,4 | (27) | □(∃x)◇~Fx | 26 □I |
| | 1,3 | (28) | ◇A → □(∃x)◇~Fx | 4,27 →I |

| | | | |
|---|---|---|---|
| | (29) | $A \to \Diamond A$ | TI |
| 1,3 | (30) | $A \to \Box(\exists x)\Diamond{\sim}Fx$ | 29,28 SI (Chain) |
| 1,2,3 | (31) | $\Box(\exists x)\Diamond{\sim}Fx$ | 30,2 $\to$E ◆ |

## Chapter 10, §2

(I.5) $S = \{\alpha,\beta,\gamma,\delta\}$, $\leqslant = \{\langle\alpha,\beta\rangle,\langle\beta,\gamma\rangle,\langle\alpha,\delta\rangle\}$, *warr*(α) = ∅, *warr*(β) = {'A'}, *warr*(γ) = {'A', 'B'}, *warr*(δ) = ∅. Since $\alpha \leqslant \beta$, $\beta \Vdash A$ and $\beta \nVdash B$, we have $\alpha \nVdash A \to B$. However, $\alpha \Vdash {\sim}B \to {\sim}A$, since $\alpha \leqslant \delta$, $\delta \Vdash {\sim}B$, $\delta \Vdash {\sim}A$, and there is no other state at which '~B' holds.

(I.12) $S = \{\alpha,\beta\}$, $\leqslant = \{\langle\alpha,\beta\rangle\}$, *warr*(α) = ∅, *warr*(β) = {'A'}. Since *warr*(α) is empty, $\alpha \nVdash A$. Since $\beta \Vdash A$ and $\beta \nVdash B$, $\alpha \nVdash A \to B$. Hence $\alpha \nVdash A \vee (A \to B)$.

## Chapter 10, §3

(6) $S = \{\alpha,\beta\}$, $\leqslant = \{\langle\alpha,\beta\rangle\}$, *dom*(α) = {❶}, *dom*(β) = {❶,❷}, *warr*(α) = ∅, *warr*(β) = {⟨F,❶⟩,⟨*G*,❷⟩}. $\alpha \nVdash (\exists x)(Fx \to Gx)$ since $\alpha \nVdash Fa \to Ga$, since $\beta \Vdash Fa$ but $\beta \nVdash Ga$. However, since $\beta \Vdash (\exists x)Fx$ and $\beta \Vdash (\exists x)Gx$, we have $\alpha \Vdash (\exists x)Fx \to (\exists x)Gx$.

(9) $S = \{\alpha,\beta\}$, $\leqslant = \{\langle\alpha,\beta\rangle\}$, *dom*(α) = {❶}, *dom*(β) = {❶,❷}, *warr*(α) = {F,❶⟩}, *warr*(β) = {F,❶⟩}. β ⊮ (∀y)(F❶ → Fy) since β ⊮ F❶ → F❷. Hence $\alpha \nVdash (\exists x)(\forall y)(Fx \to Fy)$.