Non-Abelian particles in a two dimensional world

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Bosons and fermions
Bosons and fermions
Bosons and fermions

L.D. Landau and E.M. Lifshitz

Quantum Mechanics
(Non-relativistic Theory)
Course of Theoretical Physics
Volume 3 Third Edition

L. D. Landau and E. M. Lifshitz
Institute of Physical Problems
USSR Academy of Sciences
Bosons and fermions

\[ \Psi \left( r_1, r_2 \right) \]
Bosons and fermions

\[ \Psi (r_2, r_1) = e^{i\theta} \Psi (r_1, r_2) \]
Bosons and fermions

\[ \Psi (r_1, r_2) = e^{i\theta} \Psi (r_2, r_1) = e^{2i\theta} \Psi (r_1, r_2) \]
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\[ e^{2i\theta} = 1 \]
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\[ e^{2i\theta} = 1 \]

\[ \theta = 0 \rightarrow \Psi(r_1, r_2) = \Psi(r_2, r_1) \]  

Bosons
Bosons and fermions

\[ \Psi (\mathbf{r}_1, \mathbf{r}_2) = e^{i\theta} \Psi (\mathbf{r}_2, \mathbf{r}_1) = e^{2i\theta} \Psi (\mathbf{r}_1, \mathbf{r}_2) \]

\[ e^{2i\theta} = 1 \]

\[ \theta = 0 \rightarrow \Psi (\mathbf{r}_1, \mathbf{r}_2) = \Psi (\mathbf{r}_2, \mathbf{r}_1) \quad \text{Bosons} \]

\[ \theta = \pi \rightarrow \Psi (\mathbf{r}_1, \mathbf{r}_2) = -\Psi (\mathbf{r}_2, \mathbf{r}_1) \quad \text{Fermions} \]
2D world: anyons
2D world: anyons

J. M. Leinaas (with J. Myrheim) 1977

F. Wilczek 1982 and on
2D world: anyons
2D world: anyons

J. M. Leinaas
(with J. Myrheim)
1977

F. Wilczek
1982 and on

time
2D world: anyons

counterclockwise braid

e^{i\theta}

time

J. M. Leinaas
(with J. Myrheim)
1977

F. Wilczek
1982 and on
2D world: anyons

J. M. Leinaas (with J. Myrheim) 1977
F. Wilczek 1982 and on

counterclockwise braid

time

\[ e^{2i\theta} \]

\[ e^{i\theta} \]
2D world: anyons

- Counterclockwise braid
- Clockwise braid

J. M. Leinaas and J. Myrheim (1977)
F. Wilczek (1982 and on)
2D world: anyons

J. M. Leinaas
(with J. Myrheim)
1977

F. Wilczek
1982 and on

counterclockwise braid

$e^{2i\theta}$

$e^{i\theta}$

clockwise braid

$e^{-i\theta}$
2D world: anyons

J. M. Leinaas (with J. Myrheim) 1977

F. Wilczek 1982 and on

clockwise braid

counterclockwise braid
2D world: anyons

3D: not possible

J. M. Leinaas (with J. Myrheim) 1977
F. Wilczek 1982 and on

counterclockwise braid
clockwise braid
**2D world: anyons**

3D: not possible

**counterclockwise braid**

\[ e^{2i\theta} \quad e^{i\theta} \]

**clockwise braid**

\[ e^{-2i\theta} \quad e^{-i\theta} \]

J. M. Leinaas
(with J. Myrheim)
1977

F. Wilczek
1982 and on
2D world: anyons

\[ \theta \text{ arbitrary} \]

\[ e^{2i\theta} \]

\[ e^{i\theta} \]

\[ e^{-2i\theta} \]

\[ e^{-i\theta} \]

counterclockwise braid

clockwise braid
2D world: “non-Abelions” (particles with non-Abelian statistics)
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N. Read, 1991
(originally with G. Moore)
2D world: “non-Abelions” (particles with non-Abelian statistics)

\[ \Psi_\alpha(r_1, r_2, r_3, r_4) \]

\[ \alpha = 1, \ldots, n \]
2D world: “non-Abelions” (particles with non-Abelian statistics)

\[ \Psi_\alpha(r_1, r_2, r_3, r_4) \rightarrow \sum_{\beta=1}^{n} U_{\alpha,\beta}^{(1,2)} \Psi_\beta(r_1, r_2, r_3, r_4) = \Psi_\alpha(r_2, r_1, r_3, r_4) \]

\( \alpha = 1, \ldots, n \)

(1,2) - permuting particles 1 and 2

N. Read, 1991
(originally with G. Moore)
2D world: “non-Abelions” (particles with non-Abelian statistics)

Matrices \( U^{(1,2)}_{\alpha,\beta} \), \( U^{(1,3)}_{\alpha,\beta} \) generally do not commute; hence non-Abelian

\[
\Psi_\alpha(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) \rightarrow \sum_{\beta=1}^{n} U^{(1,2)}_{\alpha,\beta} \Psi_\beta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = \Psi_\alpha(\mathbf{r}_2, \mathbf{r}_1, \mathbf{r}_3, \mathbf{r}_4)
\]

\( \alpha = 1, \ldots, n \)

(1,2) - permuting particles 1 and 2
What are anyons good for?
What are anyons good for?

A. Kitaev, 1997

Don’t know about anyons, but non-Abelions are good for the “topologically protected quantum computing”!
What are anyons good for?

Quantum bit - qubit

\[
\begin{pmatrix}
\psi_\uparrow \\
\psi_\downarrow
\end{pmatrix}
\]
What are anyons good for?

Quantum bit - qubit

\[
\begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} \rightarrow U \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = \sum_{\beta=\uparrow,\downarrow} U_{\alpha,\beta} \psi_\beta
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Decoherence - the enemy of quantum computing
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I told you so!

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Decoherence - the enemy of quantum computing

\[
\Psi_{\alpha}(r_1, r_2, r_3, r_4) \rightarrow \sum_{\beta=1}^{n} U^{(1,2)}_{\alpha,\beta} \Psi_{\beta}(r_1, r_2, r_3, r_4)
\]

random noise is powerless and decoherence is absent!

A. Kitaev, 1997
Who is interested in topological quantum computing?

One proponent is familiar to all of us...
Who is interested in topological quantum computing?

One proponent is familiar to all of us...

Bill Gates
Who is interested in topological quantum computing?

One proponent is familiar to all of us...

Bill Gates

Welcome to Station Q

Welcome!
People
Research

Station Q is a Microsoft research group working on topological quantum computing. The group combines researchers from math, physics and computer science.
Who is interested in topological quantum computing?

One proponent is familiar to all of us...

Bill Gates
Where can we find anyons?

They were actually found in the studies of fractional quantum Hall effect!
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R. B. Laughlin

H. L. Störmer

D. C. Tsui

Nobel Prize 1998
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Nobel Prize 1998

“for their discovery of a new form of quantum fluid with fractionally charged excitations”
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They were actually found in the studies of fractional quantum Hall effect!

R. B. Laughlin  
H. L. Störmer  
D. C. Tsui

Nobel Prize 1998

“for their discovery of a new form of quantum fluid with fractionally charged excitations”

Dirty little secret: those “fractionally charged excitations” are actually anyons!
Quantum Hall Effect

Hall effect (19th century): \( V = R_H I \)

magnetic field \( B \)

current
Quantum Hall Effect

Hall effect (19th century): \( V = R_H I \)

Quantum Hall effect (1980):

\[
R_H = \frac{h}{e^2 p} \quad \text{integers}
\]

Klaus von Klitzing

Nobel Prize 1985: for the discovery of QHE
Fractional Quantum Hall effect
Fractional Quantum Hall effect
Fractional Quantum Hall effect
Fractional Quantum Hall effect
Fractional Quantum Hall effect

- Magnetic field $B$
- Electrons

Diagram shows a 2D space with $x$ and $y$ axes, and a field $B$ acting perpendicular to the plane.
Fractional Quantum Hall effect

\[ z = x + iy \]

\[ \psi_n(z) = z^n e^{-\frac{|z|^2}{4\ell^2}} \]

\[ \ell = \sqrt{\frac{hc}{eB}} \]

magnetic length

single particle wave functions forming the degenerate Landau level

electrons

magnetic field B
Fractional Quantum Hall effect

\( z = x + iy \)

Single particle wave functions forming the degenerate Landau level

Arbitrary many-body wave function (antisymmetrized product of single particle wave functions)

\[
\psi(z_1, z_2, \ldots) = A(z_1, z_2, \ldots) \ e^{-\frac{1}{4\ell^2} \sum_k |z_k|^2}
\]
Anyons in fractional quantum Hall effect

Laughlin’s insight: simplest possible ground state

\[ \psi_0(z_1, z_2, \ldots) = \prod_{l < m} (z_l - z_m)^3 \ e^{-\frac{1}{4\ell^2} \sum_k |z_k|^2} \]

simplest possible excited state

\[ \psi_\eta(z_1, z_2, \ldots) = \prod_k (\eta - z_k) \ \prod_{l < m} (z_l - z_m)^3 \ e^{-\frac{1}{4\ell^2} \sum_k |z_k|^2} \]

Arbitrary many-body wave function

(antisymmetrized product of single particle wave functions)

\[ \psi(z_1, z_2, \ldots) = A(z_1, z_2, \ldots) \ e^{-\frac{1}{4\ell^2} \sum_k |z_k|^2} \]

electrons’ coordinates

arbitrary antisymmetric polynomials
Anyons in fractional quantum Hall effect

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Laughlin’s quasi-hole
Anyons in fractional quantum Hall effect

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Higher excited state (two quasiholes)

\[ \psi_{\eta_1,\eta_2}(z_1, z_2, \ldots) = (\eta_1 - \eta_2)^{\frac{1}{3}} \prod_k (\eta_1 - z_k) \prod_k (\eta_2 - z_k) \prod_{l < m} (z_l - z_m)^3 e^{-\frac{1}{4\ell^2} \sum_k |z_k|^2} \]
Anyons in fractional quantum Hall effect

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Look: those guys are anyons!
Anyons in fractional quantum Hall effect

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Look: those guys are anyons!

The normalization integral is the partition function of a 2D plasma!

$$\text{norm} = \int \prod_k d^2 z_k |\psi|^2 = \int \prod_k d^2 z_k \exp \left( \frac{2}{3} \ln |\eta_1 - \eta_2| + 2 \sum_k \ln |\eta_1 - z_k| + 2 \sum_k \ln |\eta_2 - z_k| + 6 \sum_{l < m} \ln |z_l - z_m| - \frac{1}{2\ell^2} \sum_k |z_k|^2 \right) = 1$$
Anyons in fractional quantum Hall effect

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\[ \psi_{\eta_1, \eta_2}(z_1, z_2, \ldots) = (\eta_1 - \eta_2)^{\frac{1}{3}} \prod_k (\eta_1 - z_k) \prod_k (\eta_2 - z_k) \prod_{l<m} (z_l - z_m)^3 e^{-\frac{1}{4\ell^2} \sum_k |z_k|^2} \]

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2D plasma: definitions

Two chargers interact logarithmically \( U_{12}(r) = -e_1 e_2 \ln(r) \)

The partition function is \( Z = \int \prod_k d^2 r_k e^{-\frac{1}{T} \sum_{j \neq l} U_{ji}(r_{ji})} \)
Conjecture: these are the correlation functions of a two dimensional scale invariant quantum field theory (in other words, of a statistical mechanical system at a point of a second order phase transition), and Laughlin’s guess is but a particular case of that, corresponding to a free field theory.

N. Read and G. Moore, 1991

\[ \psi(z_1, z_2, \ldots) = A(z_1, z_2, \ldots) e^{-\frac{1}{4\ell^2} \sum_k |z_k|^2} \]
Non-Abelions in fractional quantum Hall effect

\[ \psi(z_1, z_2, \ldots) = A(z_1, z_2, \ldots) e^{-\frac{1}{4\ell^2} \sum_k |z_k|^2} \]

Moore and Read: let’s take the simplest two-dimensional critical model: 2D Ising model!
Non-Abelions in fractional quantum Hall effect

\[ \psi(z_1, z_2, \ldots) = A(z_1, z_2, \ldots) e^{-\frac{1}{4\ell^2} \sum_k |z_k|^2} \]

Ising model leads to the Pfaffian (Moore-Read) state, accepted to be one of the observed quantum Hall states.
Non-Abelions in fractional quantum Hall effect

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C. Nayak, 1996 (with F. Wilczek)
Non-Abelions in fractional quantum Hall effect

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Ising model leads to the Pfaffian (Moore-Read) state, accepted to be one of the observed quantum Hall states.

\[ A_1 = \frac{[(\eta_1 - \eta_3)(\eta_2 - \eta_4)]^{\frac{1}{2}}}{\sqrt{1 - \sqrt{\frac{(\eta_2 - \eta_3)(\eta_1 - \eta_4)}{(\eta_1 - \eta_3)(\eta_2 - \eta_4)}}}} \left\{ P(\eta_1, \eta_3, \eta_2, \eta_4; z_1, z_2, \ldots) - \sqrt{\frac{(\eta_2 - \eta_3)(\eta_1 - \eta_4)}{(\eta_1 - \eta_3)(\eta_2 - \eta_4)}} P(\eta_1, \eta_4, \eta_2, \eta_3; z_1, z_2, \ldots) \right\} \]

Polynomials

\[ \eta_1 \quad \eta_2 \quad \eta_3 \quad \eta_4 \]
Non-Abelions in fractional quantum Hall effect

$$\psi(z_1, z_2, \ldots) = A(z_1, z_2, \ldots) e^{-\frac{1}{4\ell^2} \sum_k |z_k|^2}$$

Ising model leads to the Pfaffian (Moore-Read) state, accepted to be one of the observed quantum Hall states.

$$A_2 = \frac{[(\eta_1 - \eta_3)(\eta_2 - \eta_4)]^{\frac{1}{2}}}{\sqrt{1 + \sqrt{\frac{(\eta_2 - \eta_3)(\eta_1 - \eta_4)}{(\eta_1 - \eta_3)(\eta_2 - \eta_4)}}}} \left\{ P(\eta_1, \eta_3, \eta_2, \eta_4; z_1, z_2, \ldots) + \sqrt{\frac{(\eta_2 - \eta_3)(\eta_1 - \eta_4)}{(\eta_1 - \eta_3)(\eta_2 - \eta_4)}} P(\eta_1, \eta_4, \eta_2, \eta_3; z_1, z_2, \ldots) \right\}$$

Polynomials
Non-Abelions in fractional quantum Hall effect

\[ \psi(z_1, z_2, \ldots) = A(z_1, z_2, \ldots) e^{-\frac{1}{4\ell^2} \sum_k |z_k|^2} \]

Ising model leads to the Pfaffian (Moore-Read) state, accepted to be one of the observed quantum Hall states.

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Why this construction works

Proved by VG, C. Nayak, 1997

by mapping these guys into a plasma
Why this construction works

\[ \psi_1(z_1, z_2, \ldots) = A_1(z_1, z_2, \ldots) e^{-\frac{1}{4\ell^2} \sum_k |z_k|^2} \]

\[ \psi_2(z_1, z_2, \ldots) = A_2(z_1, z_2, \ldots) e^{-\frac{1}{4\ell^2} \sum_k |z_k|^2} \]

\[ A_1 = \frac{[(\eta_1 - \eta_3)(\eta_2 - \eta_4)]^{\frac{1}{4}}}{\sqrt{1 - \sqrt{\frac{(\eta_2 - \eta_3)(\eta_1 - \eta_4)}{(\eta_1 - \eta_3)(\eta_2 - \eta_4)}}}} \begin{align*}
& \left\{ P(\eta_1, \eta_3, \eta_2, \eta_4; z_1, z_2, \ldots) - \sqrt{\frac{(\eta_2 - \eta_3)(\eta_1 - \eta_4)}{(\eta_1 - \eta_3)(\eta_2 - \eta_4)}} P(\eta_1, \eta_4, \eta_2, \eta_3; z_1, z_2, \ldots) \right\} \\
& \left\{ P(\eta_1, \eta_3, \eta_2, \eta_4; z_1, z_2, \ldots) + \sqrt{\frac{(\eta_2 - \eta_3)(\eta_1 - \eta_4)}{(\eta_1 - \eta_3)(\eta_2 - \eta_4)}} P(\eta_1, \eta_4, \eta_2, \eta_3; z_1, z_2, \ldots) \right\} \end{align*} \]

\[ A_2 = \frac{[(\eta_1 - \eta_3)(\eta_2 - \eta_4)]^{\frac{1}{4}}}{\sqrt{1 + \sqrt{\frac{(\eta_2 - \eta_3)(\eta_1 - \eta_4)}{(\eta_1 - \eta_3)(\eta_2 - \eta_4)}}}} \]

Conjectured degenerate wave functions
Why this construction works

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Conjectured degenerate wave functions

where did these come from?

Proved by VG, C. Nayak, 1997 by mapping these guys into a plasma
Why this construction works

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Proven by VG, C. Nayak, 1997 and 2009

Conjectured degenerate wave functions

\[ A_1 = \frac{\sqrt{(\eta_1 - \eta_3)(\eta_2 - \eta_4)}}{\sqrt{1 - \sqrt{\frac{(\eta_2 - \eta_3)(\eta_1 - \eta_4)}{(\eta_1 - \eta_3)(\eta_2 - \eta_4)}}}} \left\{ P(\eta_1, \eta_3, \eta_2, \eta_4; z_1, z_2, \ldots) - \sqrt\frac{(\eta_2 - \eta_3)(\eta_1 - \eta_4)}{(\eta_1 - \eta_3)(\eta_2 - \eta_4)} P(\eta_1, \eta_4, \eta_2, \eta_3; z_1, z_2, \ldots) \right\} \]

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where did these come from?

Need to prove that \( \int \prod_k d^2 z_k \, \psi_\alpha^* \psi_\beta = \delta_{\alpha,\beta} \)

Proven by VG, C. Nayak, 1997 and 2009
Status of the Non-Abelions in FQHE

• Overwhelming numerical evidence that the non-Abelian quantum Hall states exist as well as firm experimental evidence that they have been observed (states were observed which, as is firmly believed, must have particles with non-Abelian statistics).

• However, nobody was able to probe the non-Abelian statistics experimentally. They see the fractional charge consistent with statistics, but not the statistics itself.

Observation of a quarter of an electron charge at the $\nu = 5/2$ quantum Hall state

M. Dolev¹, M. Heiblum¹, V. Umansky¹, Ady Stern¹ & D. Mahalu¹
Question: can we look for the non-Abelian particles elsewhere?
Topological states of matter

X.-G. Wen
Topological states of matter

2D states of matter with fractional and/or non-Abelian excitations

X.-G. Wen
Topological states of matter

Examples realized or potentially realizable in nature:

1. Fractional Quantum Hall Effect. It’s observed and is surely topological. Attempts to observe its non-Abelian particles were so far not successful. More work is ongoing.
Topological states of matter

Examples realized or potentially realizable in nature:

1. **Fractional Quantum Hall Effect.** It’s observed and is surely topological. Attempts to observe its non-Abelian particles were so far not successful. More work is ongoing.

2. **2D $p_x + i p_y$ superconductors.** Sr$_2$RuO$_4$ is the most promising candidate, but no unambiguous evidence.

Proposal to realize such superconductor using cold atoms, VG, A. Andreev, and L. Radzihovsky (2004-05).
Topological states of matter

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3. **Chiral spin liquids.** Long sought after topological state of quantum magnets.

   Proposal to realize it using cold atoms, M. Hermele, VG, Ana-Maria Rey (2009).
Superconductivity

Superconductors:
- conduct electricity without any resistance;
- expel magnetic fields (Meissner effect), levitate in a mag field;
- are Bose-condensates of pairs of electrons, “Cooper pairs”;
- form when electrons experience attraction;

Kamerlingh Onnes
1911
Nobel Prize 1913
Excitations in a superconductor

Bogoliubov quasiparticles
Excitations in a superconductor

Bogoliubov quasiparticles

Quasiparticle annihilation

\[ \hat{\gamma}_n = \int dr \ [ u_n(r) \hat{a}(r) + v_n(r) \hat{a}^\dagger(r) ] \]

and

creation operators

\[ \hat{\gamma}^\dagger_n = \int dr \ [ u^*_n(r) \hat{a}^\dagger(r) + v^*_n(r) \hat{a}(r) ] \]
Excitations in a superconductor

Bogoliubov quasiparticles

Quasiparticle wavefunctions

Quasiparticle annihilation

\[ \hat{\gamma}_n = \int d\mathbf{r} \left[ u_n(\mathbf{r}) \hat{a}(\mathbf{r}) + v_n(\mathbf{r}) \hat{a}^\dagger(\mathbf{r}) \right] \]

electron’s annihilation and creation operator

and creation operators

\[ \hat{\gamma}_n^\dagger = \int d\mathbf{r} \left[ u_n^*(\mathbf{r}) \hat{a}^\dagger(\mathbf{r}) + v_n^*(\mathbf{r}) \hat{a}(\mathbf{r}) \right] \]
Excitations in a superconductor

Bogoliubov quasiparticles

Quasiparticle annihilation \[ \hat{\gamma}_n = \int dr \left[ u_n(r) \hat{a}(r) + v_n(r) \hat{a}^\dagger(r) \right] \]

and quasiparticle creation operators \[ \hat{\gamma}_n^\dagger = \int dr \left[ u_n^*(r) \hat{a}^\dagger(r) + v_n^*(r) \hat{a}(r) \right] \]

What if: \[ v_n = u_n^* \] (for some \( n \))

Not a creation operator of anything...
Excitations in a superconductor

Bogoliubov quasiparticles

Quasiparticle wavefunctions

Quasiparticle annihilation

\[ \hat{\gamma}_n = \int \, dr \, \left[ u_n(r) \, \hat{a}(r) + v_n(r) \, \hat{a}^\dagger(r) \right] \]

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\[ \hat{\gamma}_n = \hat{\gamma}_n^\dagger = \int \, dr \, \left[ u_n(r) \, \hat{a}(r) + u_n^*(r) \, \hat{a}^\dagger(r) \right] \]

Not a creation operator of anything...

\[ \hat{c} = \hat{\gamma}_1 + i\hat{\gamma}_2 \]

These are legitimate creation and annihilation operators

\[ \hat{c}^\dagger = \hat{\gamma}_1 - i\hat{\gamma}_2 \]

Each of these \( \gamma \) are half of the electron! (an anyon, isn’t it??)
Excitations in a 2D $p_x + i p_y$ superconductor

This is a 2D superconductor where Cooper pairs of electrons spin about their center of mass with angular momentum $1$ (p-wave) and with $\ell_z = 1$ ($p_x + i p_y$).
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\[
\hat{c} = \hat{\gamma}_1 + i \hat{\gamma}_2 \\
\hat{c}^\dagger = \hat{\gamma}_1 - i \hat{\gamma}_2
\]

\[
\hat{\gamma}_n = \hat{\gamma}_n^\dagger = \int d\mathbf{r} \left[ u_n(\mathbf{r}) \hat{a}(\mathbf{r}) + u_n^*(\mathbf{r}) \hat{a}^\dagger(\mathbf{r}) \right]
\]

It's been shown that they realize the fractionalization scenario!

Detailed studies showed these are non-Abelions, just as in Quantum Hall Effect

Volovik, 1990s
Kopnin, Salomaa, 1991
N. Read, D. Green, 2000
D. Ivanov, 2001
A. Stern et al, 2002-
VG and L. Radzihovsky, 2007
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$$\hat{c} = \hat{\gamma}_1 + i\hat{\gamma}_2$$
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$$u_1(r)$$
$$u_2(r)$$

$$\hat{\gamma}_1$$
$$\hat{\gamma}_2$$

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Where can we find such a superconductor?

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Cold atoms to the rescue?
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- Take atoms with desired preselected interactions, mix them together and simulate any many-body state of nature
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The promise of cold atoms:

• Take atoms with desired preselected interactions, mix them together and simulate any many-body state of nature

The drawbacks of cold atoms:

• Not all interactions can be modeled. Atoms are neutral, so magnetic fields are hard to emulate. Coulomb or other long range interactions are hard as well

• Cold atom systems often tend to be unstable, especially those with interesting interactions
Superconductivity with cold atoms

Take a bunch of fermionic atoms (common examples $^{40}$K or $^6$Li), “turn on” attractive interactions between them, cool them down, and they form a superconductor!
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D. Jin, M. Greiner, C. Regal, ‘03-04
Superconductivity with cold atoms

Take a bunch of fermionic atoms (common examples $^{40}\text{K}$ or $^{6}\text{Li}$), “turn on” attractive interactions between them, cool them down, and they form a superconductor!

$^{40}\text{K}, F_z=-9/2$

$^{40}\text{K}, F_z=-7/2$

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Superconductivity with cold atoms

Take a bunch of fermionic atoms (common examples $^{40}$K or $^{6}$Li), “turn on” attractive interactions between them, cool them down, and they form a superconductor!

s-wave Cooper pairs

D. Jin, M. Greiner, C. Regal, ‘03-04
p-wave superconductors with cold atoms

VG, A. Andreev, L. Radzihovsky, 04-05
Observation 1: identical fermionic atoms form Cooper pairs with odd angular momentum. For example, $L=1$. 

Atoms in the same state - identical
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Observation 2: it is energetically favorable for the Cooper pairs to have \( \ell_z = 1 \) (to verify this requires a many-body calculation)
**p-wave superconductors with cold atoms**

Observation 1: identical fermionic atoms form Cooper pairs with odd angular momentum. For example, $L=1$.

Observation 2: it is energetically favorable for the Cooper pairs to have $\ell_z = 1$ (to verify this requires a many-body calculation).

Observation 3: take identical fermionic atoms, cool them down, confine them to 2D, turn on attractive interactions, and you will get a 2D $p_x + ip_y$ superconductor.

VG, A. Andreev, L. Radzihovsky, 04-05
Experiments

PRL 98, 200403 (2007)

p-Wave Feshbach Molecules

J. P. Gaebler, * J. T. Stewart, J. L. Bohn, and D. S. Jin

JILA, Quantum Physics Division, National Institute of Standards and Technology and Department of Physics, University of Colorado, Boulder, Colorado 80309-0440, USA

(Received 2 March 2007; published 16 May 2007)

Bottom line:
the molecules are unstable, with $\tau \sim 2\text{ms}$
Origin of instability: 3 body recombination

$R_e \sim 25 - 50 \text{ a.u.}$

$R_e$ is the so-called van der Waals length (the typical interaction range)
Origin of instability: 3 body recombination

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Protected by the Pauli principle
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By having opposite angular momenta, $p$-wave fermions beat the Pauli principle.

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**s-wave**

Protected by the Pauli principle

**p-wave**

By having opposite angular momenta, $p$-wave fermions beat the Pauli principle
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Origin of instability: 3 body recombination

Protected by the Pauli principle:

- \( s \)-wave

By having opposite angular momenta, \( p \)-wave fermions beat the Pauli principle.
Lifetime calculations

Interatomic distance

atomic mass

\[ \text{Lifetime} = \frac{mr^2}{\hbar} \frac{r}{R_e} \approx 20\text{ms} \]

van der Waals length

Probably, their life is too short!

J. Levinsen, N. Cooper, VG, 07-08
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J. Levinsen, N. Cooper, VG, 07-08

Optical lattices may provide a way to overcome short lifetimes...

P. Zoller et al, 09
topological magnets
topological magnets

X.-G. Wen  F. Wilczek  A. Zee

1989
topological magnets

Heisenberg antiferromagnet

\[ H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \]

Nearest neighbors

Néel state
topological magnets

Heisenberg antiferromagnet

\[ H = J \sum_{(ij)} \vec{S}_i \cdot \vec{S}_j \]

Nearest neighbors

Néel state

Chiral spin liquid (CSL)

Think of spin as attached to particles

\[ \uparrow \quad \uparrow \]

spin-up spin-down

\[ f_{i \uparrow}^{\dagger}, f_{i \uparrow}; f_{i \downarrow}^{\dagger}, f_{i \downarrow} \]

\[ H = J \sum_{<ij>, \alpha, \beta = \uparrow, \downarrow} f_{i, \alpha}^{\dagger} f_{i, \beta} f_{j, \beta}^{\dagger} f_{j, \alpha} \]
topological magnets

Heisenberg antiferromagnet

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Nearest neighbors

Néel state

Chiral spin liquid (CSL)

Think of spin as attached to particles

spin-up \hspace{1cm} spin-down

\[ H = J \sum_{\langle ij \rangle, \alpha, \beta=\uparrow, \downarrow} \langle f_i^{\dagger}, f_i, \beta | f_j, \beta \rangle f_j, \alpha \]

What if \[ \sum_{\alpha} \langle f_i^{\dagger}, \alpha | f_j, \alpha \rangle = t_{ij} \]

\[ H = J \sum_{\langle ij \rangle, \beta} t_{ij} f_i^{\dagger}, \beta f_j, \beta + \ldots \]

“tight-binding Hamiltonian”
topological magnets

Heisenberg antiferromagnet

\[ H = J \sum_{<ij>} \vec{S}_i \cdot \vec{S}_j \]

Néel state

Chiral spin liquid (CSL)

Think of spin as attached to particles

\[ f_{i\uparrow}, f_{i\uparrow}; f_{i\downarrow}, f_{i\downarrow} \]

spin-up, spin-down

\[ H = J \sum_{<ij>,\alpha,\beta=\uparrow,\downarrow} \left( f_{i,\alpha} \right)^\dagger f_{j,\beta} f_{j,\beta} \left( f_{i,\alpha} \right)^\dagger \]

What if \( \sum_{\alpha} \left( f_{i,\alpha} \right)^\dagger f_{j,\alpha} = t_{ij} \)

\[ H = J \sum_{<ij>,\alpha,\beta} t_{ij} f_{i,\alpha} \left( f_{j,\beta} \right)^\dagger \]

“tight-binding Hamiltonian”

But what if \( t_{ij} \) correspond to a constant magnetic field?

This is CSL (or a topological magnet), by analogy with QHE.
20 years and 552 citations later, nobody could still point out the Hamiltonian for which this scenario would work.
A proposal to generalize spin from SU(2) to SU(N)

Generalize the usual spin to SU(N) spin by using alkaline-earth atoms. Their nuclear spin does not interact and behaves like an electron spin, only larger.

The spin $I$ can be as large as $9/2$ (for $^{87}\text{Sr}$). Then $N=2I+1$ is as large as 10.

A.-M. Rey (2009)
SU(N) antiferromagnets in optical lattices

Interfering laser beams
SU(N) antiferromagnets in optical lattices

$^{87}$Sr atoms
SU(N) antiferromagnets in optical lattices

Atom exchange leads to antiferromagnetic interactions (for nuclear spin).

\[ H = J \sum_{\langle ij \rangle, \alpha, \beta=1, \ldots, N} f^\dagger_{i,\alpha} f_{i,\beta} f^\dagger_{j,\beta} f_{j,\alpha} \]
SU(N) antiferromagnets in optical lattices

Atom exchange leads to antiferromagnetic interactions (for nuclear spin).

\[ H = J \sum_{<ij>, \alpha, \beta=1,\ldots,N} f_{i,\alpha}^\dagger f_{i,\beta} f_{j,\beta}^\dagger f_{j,\alpha} \]

Such SU(N) spins have a hard time ordering: too many directions nearby spins can point to while still being “opposite” to each other (minimize \( \vec{S}_i \cdot \vec{S}_j \))

M. Hermele (2009)
It turns out, for $N \geq 5$, the ground state is a chiral spin liquid (that is, a topological magnet), exactly of the type proposed by Wen, Wilczek and Zee.

M. Hermele, VG, A.-M. Rey, (2009)
Topological SU(N) antiferromagnet

It turns out, for $N \geq 5$, the ground state is a chiral spin liquid (that is, a topological magnet), exactly of the type proposed by Wen, Wilczek and Zee.

M. Hermele, VG, A.-M. Rey, (2009)

To show that, we employed the large $N$ techniques:

$$H = J \sum_{i,\alpha} t_{ij} \left( f_{i,\alpha}^\dagger f_{j,\alpha} + h c \right) + \frac{N}{J} \sum_{\langle ij \rangle} |t_{ij}|^2$$

$$S = N \text{Tr} \log [S_{ij}] + \frac{N}{J} \sum_{\langle ij \rangle} |t_{ij}|^2$$

+ saddle point in $t$
Anyons and non-Abelions

Lowering the potential at one site localizes a fractional or non-Abelian particle at that site.
Anyons and non-Abelions

Lowering the potential at one site localizes a fractional or non-Abelian particle at that site.

Experimental detection? Too soon to tell...
Conclusions and outlook

Non-Abelian particles:

• definitely exist, but have not yet been seen

• would be very exciting to find, both for fundamental and applied reasons

• have excellent prospects of being found

• are a wonderful playground for a theorist
The end.