Logarithmic operators at c=0

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Thanks to:
Alexander Polyakov (1993)
Chetan Nayak and Michael Flohr (1996-1997)
Alexei Tsvelik and Ian Kogan (1995-2000)
Andreas Ludwig and Monwhea Jeng (1998-2004)

Logarithmic operators

Logarithmic pair *C*, *D*

$$L_0 \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}$$

Correlation functions consistent with (and following from) the Jordan block structure for L_0

$$\langle C(z)\, C(w) \rangle = 0 \longrightarrow$$

C must have zero norm

$$\langle C(z)\, D(w) \rangle = \frac{1}{(z-w)^{2\lambda}} \longrightarrow$$

C and D correlate like "normal" operators

$$\langle D(z)\, D(w) \rangle = -\frac{2\ln(z-w)}{(z-w)^{2\lambda}} \rightarrow$$

-2 is enforced by conformal invariance

Self-avoiding random walks (SARW)

Wiener-Feynman, 30-40s:

The probability of observing a particle undergoing Brownian motion at a point *x* at a time *t*

$$P(t,x) = \int_{x(0)=0}^{x(t)=x} \mathcal{D}x(t)\, e^{-\frac{1}{D}\int_0^t dt\, \dot{x}_\mu^2}$$

$$P(t,x) \sim e^{-\frac{x^2}{Dt}} \qquad \langle x^2 \rangle \sim Dt$$

SARW/polymers: Polymers are penalized energetically when they intersect themselves (Flory, de Gennes & others, 60s-70s)

$$P(t,x) = \int_{x(0)=0}^{x(t)=x} \mathcal{D}x(t)\, e^{-\frac{1}{D}\int_0^t dt\, \dot{x}_\mu^2 - \frac{g}{2}\int dt dt'\, \delta\left(\vec{x}(t)-\vec{x}'(t)\right)}$$

Hard to solve, but the following scaling ansatz helps

$$P(t,x) \sim \frac{e^{-i\omega_c t}}{t x^\eta} \tilde{f}\left(\frac{x^{\frac{2-\eta}{\gamma}}}{t}\right) \longleftrightarrow P(\omega,k) \sim \frac{1}{k^{2-\eta}} f\left(\frac{\omega-\omega_c}{k^{\frac{2-\eta}{\gamma}}}\right)$$

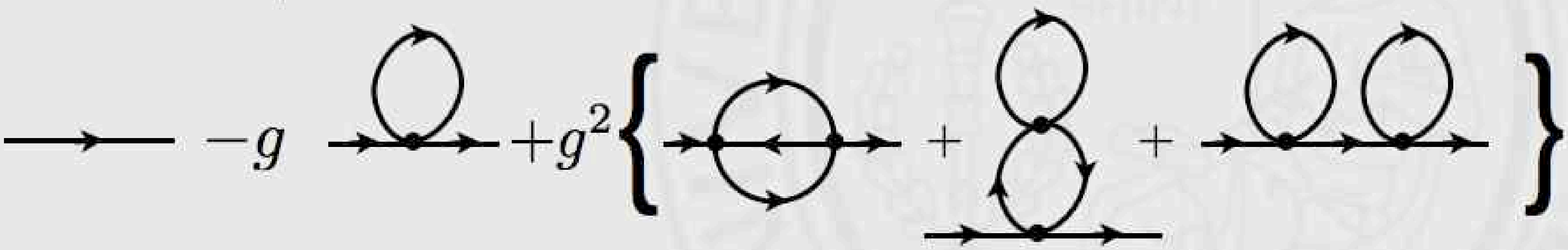
$$\langle x^2 \rangle \sim t^{\frac{2\gamma}{2-\eta}}$$

These are messy details, but the bottom line is clear: P(t,x) is some sort of a Green's function of an interacting critical theory, with ω (Fourier of *t*) a relevant perturbation

SARW: Effective field theory

$$P(t,x) = \int_{x(0)=0}^{x(t)=x} \mathcal{D}x(t)\, e^{-\frac{1}{D}\int_0^t dt\, \dot{x}_\mu^2 - \frac{g}{2}\int dt dt'\, \delta\left(\vec{x}(t)-\vec{x}'(t)\right)}$$

Perturbative expansion



is reproduced by the expansion of this Green's function with a random imaginary potential *i V(x)* in powers of *V(x)*

$$\frac{1}{i\omega + D\frac{\partial^2}{\partial x^2} - iV(x)} \qquad \langle V(x)V(y)\rangle = g\,\delta(x-y)$$

$$P(\omega,x) = \frac{\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\,\phi(x)\,\bar{\phi}(0)\, e^{\int d^2x\,\bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}{\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\, e^{\int d^2x\,\bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}$$

Random potentials: replica approach

$$P(\omega, x) = \frac{\int \mathcal{D}\phi \mathcal{D}\bar{\phi}\, \phi(x)\, \bar{\phi}(0)\, e^{\int d^2x\, \bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}{\int \mathcal{D}\phi \mathcal{D}\bar{\phi}\, e^{\int d^2x\, \bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}$$

Random potential

$$\langle V(x)V(y)\rangle = g\,\delta(x-y)$$

Introduce *n* replicas

$$P(\omega, x) = \frac{\int \prod_{i=1}^n \mathcal{D}\phi_i \mathcal{D}\bar{\phi}_i\, \phi_1(x)\, \bar{\phi}_1(0)\, e^{\sum_{i=1}^n \int d^2x\, \bar{\phi}_i\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi_i}}{\left[\int \mathcal{D}\phi \mathcal{D}\bar{\phi}\, e^{\int d^2x\, \bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}\right]^n}$$

take *n* to zero

$$P(\omega, x) = \lim_{n\to 0} \int \prod_{i=1}^n \mathcal{D}\phi_i \mathcal{D}\bar{\phi}_i\, \phi_1(x)\, \bar{\phi}_1(0)\, e^{\sum_{i=1}^n \int d^2x\, \bar{\phi}_i\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi_i}$$

and finally average over random potential

$$P(\omega, x) = \lim_{n\to 0} \int \prod_{i=1}^n \mathcal{D}\phi_i \mathcal{D}\bar{\phi}_i\, \phi_1(x)\, \bar{\phi}_1(0)\, e^{-\boxed{\int d^2x \left[\sum_{i=1}^n D\partial_\mu\bar{\phi}_i\partial_\mu\phi_i - i\omega\bar{\phi}_i\phi_i + \frac{g}{2}\left(\sum_{i=1}^n \bar{\phi}_i\phi_i\right)^2\right]}}$$

This is the famous *O(n)* model in the $n \to 0$ limit

Random potentials: "supersymmetry approach"

$$P(\omega, x) = \frac{\int \mathcal{D}\phi \mathcal{D}\bar{\phi}\, \phi(x)\, \bar{\phi}(0)\, e^{\int d^2x\, \bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}{\int \mathcal{D}\phi \mathcal{D}\bar{\phi}\, e^{\int d^2x\, \bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}} \qquad \langle V(x) V(y)\rangle = g\,\delta(x-y)$$

Introduce fermionic fields ψ

$$P(\omega, x) = \int \mathcal{D}\phi \mathcal{D}\bar{\phi} \mathcal{D}\bar{\psi} \mathcal{D}\psi\, \phi(x)\, \bar{\phi}(0)\, e^{\int d^2x\, \left[\bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi + \bar{\psi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\psi\right]}$$

Average over random potential, to find effective field theory with the action

$$S = \int d^2x \left[D\left(\partial_\mu \bar{\phi}\, \partial_\mu \phi + \partial_\mu \bar{\psi}\, \partial_\mu \psi\right) + \frac{g}{2}\left(\bar{\phi}\phi + \bar{\psi}\psi\right)^2 \right]$$

<u>We would like to study CFTs corresponding to the field theories of this type. All have c=0.</u>

“Supersymmetric” critical theories

- Supersymmetric effective field theories describe a variety of interesting critical behavior in 2 dimensions. Most have not been understood.

- Examples include self-avoiding random walks and percolation (mostly understood, although not completely) and quantum motion in random potentials under various conditions (mostly not understood).

- Most famous example, the quantum Hall transition, has been extensively studied, and yet is not understood.

Supersymmetry

A typical action

$$S = \int d^2x \left[D \left(\partial_\mu \bar{\phi}\, \partial_\mu \phi + \partial_\mu \bar{\psi}\, \partial_\mu \psi \right) + \frac{g}{2} \left(\bar{\phi}\phi + \bar{\psi}\psi \right)^2 \right]$$

$$\begin{pmatrix} \phi' \\ \psi' \end{pmatrix} = \begin{pmatrix} \alpha_1 & \epsilon \\ \bar{\epsilon} & \alpha_2 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

Superunitary (more precisely, in this example, orthosymplectic) group is the symmetry group of this action

<u>Strange reducible but indecomposable representations of the superunitary group</u>

$$\begin{tikzcd}
 & \bar{\phi}\phi - \bar{\psi}\psi \arrow[dl, "\epsilon"'] \arrow[dr, "\bar{\epsilon}"] & \\
\bar{\phi}\psi \arrow[dr, "\bar{\epsilon}"'] & & \phi\bar{\psi} \arrow[dl, "\epsilon"] \\
 & \bar{\phi}\phi + \bar{\psi}\psi &
\end{tikzcd}$$

scalar at the bottom

Logarithms and the indecomposable reps

$$
\begin{tikzcd}
 & D \arrow[dl] \arrow[dr] & \\
\zeta \arrow[dr] & & \bar{\zeta} \arrow[dl] \\
 & C &
\end{tikzcd}
$$

Logarithmic operators love indecomposable multiplets

Z. Masarani, D. Serban, 1996

$\langle C(z)\, C(w) \rangle = 0$ Used to be mysterious, now natural $\delta \left\langle \zeta(z) C(w) \right\rangle = 0$

$$\langle C(z)\, D(w) \rangle = \frac{1}{(z-w)^{2\lambda}}$$

$$\delta \left\langle D(z) \bar{\zeta}(w) \right\rangle = \left\langle \zeta(z) \bar{\zeta}(w) \right\rangle - \left\langle D(z) C(w) \right\rangle = 0$$

So ζ are just usual primary fields

$$\left\langle \zeta(z) \bar{\zeta}(w) \right\rangle = \frac{1}{(z-w)^{2\lambda}}$$

Finally:

$$\langle D(z)\, D(w) \rangle = -\frac{2 \ln(z-w)}{(z-w)^{2\lambda}}$$

because why not??

Stress-energy tensor at c=0: CFT perspective

Any primary operator with a nonvanishing norm in a CFT satisfies

$$A(z)A(0) = \frac{1}{z^{2\lambda}}\left(1 + \frac{2\lambda}{c}T(z) + \dots\right)$$

Thus the direct limit *c*→*0* is problematic.

Any *c=0* CFT must contain operators with dimension 2 distinct from the stress-energy tensor. At least one of them, called *t*, must satisfy

$$T(z)t(0) = \frac{b}{z^4} + \dots$$

Then $A(z)A(0) = \frac{1}{z^{2\lambda}}\left(1 + \frac{\lambda}{b}t(z) + CT(0) + \dots\right)$

VG, 1999

Stress-energy tensor at c=0: supersymmetry perspective

$$\begin{tikzcd} & t \arrow[dl] \arrow[dr] & \\ \xi \arrow[dr] & & \bar{\xi} \arrow[dl] \\ & T & \end{tikzcd}$$

Stress-energy tensor is always a part of a reducible but indecomposable multiplet

Possible consistent OPE:

$$T(z)T(0) = \frac{2T(0)}{z^2} + \dots$$

$$T(z)t(0) = \frac{b}{z^4} + \frac{2t(0)}{z^2} + \dots$$

$$t(z)t(0) = \frac{2t(0)}{z^2} + \dots$$

Realized in supergroup-based WZW models.

But these are also possible consistent OPE:

$$T(z)T(0) = \frac{2T(0)}{z^2} + \dots$$

$$T(z)t(0) = \frac{b}{z^4} + \frac{2t(0) + T(0)}{z^2} + \dots$$

$$t(z)t(0) = \frac{-2b \ln z}{z^4} + \dots$$

Makes *t* logarithmic. Realized in *c=0* minimal model.

Nonlogarithmic t: free field theory

$$S \sim \int d^2x \left[\left(\partial_\mu \bar{\phi}\, \partial_\mu \phi + \partial_\mu \bar{\psi}\, \partial_\mu \psi \right) \right]$$

Bosons Fermions

Stress-tensor multiplet

$$T = \partial\bar{\phi}\partial\phi + \partial\bar{\psi}\partial\psi$$

$$t = \partial\bar{\phi}\partial\phi - \partial\bar{\psi}\partial\psi$$

$$\xi = \partial\bar{\phi}\partial\psi$$

$$T(z)T(0) = \frac{2T(0)}{z^2} + \ldots$$

$$T(z)t(0) = \frac{b}{z^4} + \frac{2t(0)}{z^2} + \ldots$$

$$t(z)t(0) = \frac{2t(0)}{z^2} + \ldots$$

$$b = 2$$

b in this case is the central charge of the bosonic part of the theory

Nonlogarithmic t: Kac-Moody algebras

U(1|1) Kac-Moody algebra with the generators $J,\ j,\ \eta,\ \bar{\eta}$

sort of like U(2), but with different:

$$[j,\ \eta] = -\eta$$
$$[j,\ \bar{\eta}] = \bar{\eta}$$
$$\{\eta,\ \bar{\eta}\} = J$$

J commutes with everybody

C. Chamon, C. Mudry, X.-G. Wen, 1996

$$T = \frac{k}{2}\left(Jj + \eta\bar{\eta} - \bar{\eta}\eta\right) + k^2 \frac{4-k_j}{8} JJ$$

$$\xi = \frac{k}{4}\left(\eta j + j\eta\right) + k\frac{4-k_j}{8}\eta J$$

$$t = \frac{k}{4} jj + k^2 \frac{4-k_j}{16}\left(Jj + \bar{\eta} - \eta\bar{\eta}\right)$$

VG, 1999

$$T(z)T(0) = \frac{2T(0)}{z^2} + \dots$$

$$T(z)t(0) = \frac{b}{z^4} + \frac{2t(0)}{z^2} + \dots$$

$$t(z)t(0) = \frac{2t(0)}{z^2} + \dots$$

$$b = k$$

Logarithmic t: supersymmetry emerges

$$T(z)t(0) = \frac{b}{z^4} + \frac{2t(0) + T(0)}{z^2} + \frac{t'(0)}{z} + \dots$$

$$t(z)t(0) = -\frac{2b\log z}{z^4} + \frac{t(0)\left[1 - 4\log z\right] - T(0)\left[\log z + 2\log^2 z\right]}{z^2}$$

$$\xi(z)\bar{\xi}(0) = \frac{1}{8}T(z)T(0) + \frac{b}{2z^4} + \frac{t(0) + T(0)\log z}{z^2} + \dots$$

$$t(z)\xi(0) = \frac{1}{4}T(z)\xi(0) - T(z)\xi(0)\log z + \frac{\xi'(0)}{2z} + \dots$$

These follow from the assumption of logarithmic t by conformal invariance only

$$\begin{tikzcd} & t \arrow[dl] \arrow[dr] & \\ \xi \arrow[dr] & & \bar{\xi} \arrow[dl] \\ & T & \end{tikzcd}$$

Yet they automatically form the indecomposable representation shown on the left

Example of a derivation

$$\xi(z)\bar{\xi}(0) = \alpha T(z)T(0) + \frac{b}{2z^4} + \frac{t(0) + T(0)\ln z}{z^2} + \dots$$

Don't know **α** at the moment.

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$

$$G = \langle \xi(z_1)\bar{\xi}(z_2)\bar{\xi}(z_3)\xi(z_4) \rangle$$ Let's compute it

This is a rational function:

$$G_{\mathrm{mod}} = \langle \xi(z_1)\bar{\xi}(z_2)\bar{\xi}(z_3)\xi(z_4) \rangle - \frac{1}{2}\langle T(z_1)T(z_2)\bar{\xi}(z_3)\xi(z_4) \rangle \ln x$$

Reconstruct it by its singularities!

$$G = \frac{1}{(z_1 - z_2)^4 (z_3 - z_4)^4} \left[\frac{(x+1)(2x^2 + b(x-1)^2(1+x^2))}{4(x-1)} - \frac{x^2(1-x+x^2)\ln x}{(1-x)^2} \right]$$

Only works if **α**=1/4.

Extended algebra?

In the same way how Virasoro algebra can be derived from the OPEs (as well as extended W-algebras), can an extended algebra of dimension 2 operators follow from the logarithmic OPEs?

The answer to this question is not known. But there exist partial examples which show that this may work.

$$T(z)t(0) = \frac{b}{z^4} + \frac{2t(0) + T(0)}{z^2} + \frac{t'(0)}{z} + \ldots$$

$$t(z)t(0) = -\frac{2b \ln z}{z^4} + \frac{t(0)\left[1 - 4\ln z\right] - T(0)\left[\ln z + 2\ln^2 z\right]}{z^2}$$

$$\xi(z)\bar{\xi}(0) = \frac{b}{2z^4} + \frac{t(0) + T(0)\ln z + \frac{1}{4}T(0)}{z^2} + \ldots$$

$$t(z)\xi(0) = \frac{\frac{1}{2}\xi(0) - 2\xi(0)\ln z}{z^2} + \frac{3\xi'(0) - 4\xi'(0)\ln z}{4z} + \ldots$$

Attempts to construct extended algebra

Conformal invariance predicts:

$$\langle T(z)A(w_1)A(w_2)\rangle = \frac{\lambda}{(z-w_1)^2(z-w_2)^2(w_1-w_2)^{2\lambda-2}}$$

$$\langle t(z)A(w_1)A(w_2)\rangle = \frac{\lambda \ln\left[\frac{w_1-w_2}{(z-w_1)(z-w_2)}\right] + \text{const}}{(z-w_1)^2(z-w_2)^2(w_1-w_2)^{2\lambda-2}}$$

$$-\log(z-w_1)\,\langle T(z)A(w_1)A(w_2)\rangle$$

We recognize that this must be true:

$$t(z)A(w) = -T(z)A(w)\log(z-w) + \text{regular stuff}$$

Logarithmic algebra

Logarithms in the OPE of t and A (A - arbitrary primary operator with nonzero dimension) can be removed:

$$t(z)A(0) = \underbrace{-T(z)A(0)\ln z}_{\text{Logarithms are captured by this term}} + \sum_{n=0}^{\infty} \ell_{-n} A(0) z^{n-2}$$

$$\ell_n A(0) = \oint \frac{dz}{2\pi i} \left(t(z) + \ln(z)T(z)\right) z^{n+1} A(0)$$

$$[\ell_n, L_m] = \oint dz dw \left(t(z) + \ln(z)T(z)\right) T(w)\, z^{n+1} w^{m+1}$$

$$[\ell_n, L_m] = \frac{b}{6} n(n^2-1)\delta_{n+m,0} + (n-m)\ell_{n+m} - mL_{n+m}$$

Logarithmic commutation relations

Generalization of these to other components of the stress tensor multiplet were not yet found.

Logarithmic t: minimal model at c=0

$$b = -\frac{5}{8}$$

n \ m	1	2	3	4	5
5	2	$\frac{5}{8}$	0		
4	1	$\frac{1}{8}$	0		
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$		
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7

$$b = \frac{5}{6}$$

$$\lambda_{m,n} = \frac{(2n-3m)^2-1}{24}$$

Differential equations give

$$\langle A(z_1)A(z_2)A(z_3)A(z_4)\rangle = \frac{1}{(z_1-z_2)^{2\lambda}(z_3-z_4)^{2\lambda}}\left(1+\alpha x^2\ln(x)+\dots\right)$$

$$\alpha = \frac{\lambda}{b} \qquad\qquad x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

Algebraic approach to compute b

$$b = -\frac{5}{8}$$

$$G = \frac{1}{(z_1 - z_2)^4 (z_3 - z_4)^4} \left[\frac{(x+1)(2x^2 + b(x-1)^2(1+x^2))}{4(x-1)} - \frac{x^2(1 - x + x^2)\ln x}{(1-x)^2} \right]$$

Satisfies appropriate equations at the appropriate values of b

n \ m	1	2	3	4	5
5	2	$\frac{5}{8}$	0		
4	1	$\frac{1}{8}$	0		
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$		
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7

$$\lambda_{m,n} = \frac{(2n - 3m)^2 - 1}{24}$$

$$b = \frac{5}{6}$$

$$\left(L_{-2} - L_{-1}^2\right)\left|\frac{5}{8}\right\rangle$$ Null vector

$$\ell_2 \left(L_{-2} - L_{-1}^2\right)\left|\frac{5}{8}\right\rangle = 0 \longrightarrow b = \frac{5}{6}$$

Monwhea Jeng: correct up to at least the degeneracy level 15

Operators with vanishing dimension

An operator of dimension 0 at c=0 which is primary and not identity plays a special role in Cardy's theory of percolation...

$$\langle T(z) O(w_1) O(w_2) \rangle = 0$$

$$\langle t(z) O(w_1) O(w_2) \rangle = \frac{\Delta (w_1 - w_2)^2}{(z - w_1)^2 (z - w_2)^2}$$

$$t(z) O(0) = -(1 - \epsilon) T(z) O(0) \ln z + \text{regular stuff}$$

$$\ell_2 \left(L_{-2} + \frac{3}{2} L_{-1} \right) |0\rangle = 0 \longrightarrow b = \frac{5(7\epsilon - 5)}{12}$$

Difficulties if one tries to go further

- Commutation relations depend on what the operators act on (but isn't it similar to the parafermions)?

- What if the operators that the stress-tensor multiplet acts on are themselves parts of multiplets?

- Substracting logarithms may or may not be possible in all the cases.

- What if gluing left and right sector is not a trivial task?

Cardy's explanation of the logarithms

$$Z = \int \exp\left[-S_0 - \int d^2x\, t(x) E(x)\right].$$

$$Z^n = \int \exp\left[-\sum_{a=1}^{n} S_{0,n} + g \int d^2x\, E_a(x) E_b(x)\right].$$

$$\langle E(x)E(0)\rangle = \lim_{n\to 0} \langle E_1(x)E_1(x)\rangle$$

$$\tilde{E} = \sum_{a=1}^{n} E_a$$

$$\tilde{E}_a = E_a - \frac{1}{n} E$$

$$\frac{1}{n}\left\langle \tilde{E}(x)\tilde{E}(0)\right\rangle = \langle E_1(x)E_1(0)\rangle + (n-1)\langle E_1(x)E_2(0)\rangle = \frac{A(n)}{x^{2\Delta(n)}}$$

$$\frac{n}{n-1}\left\langle \tilde{E}_a(x)\tilde{E}_a(0)\right\rangle = \langle E_1(x)E_1(0)\rangle - \langle E_1(x)E_2(0)\rangle = \frac{B(n)}{x^{2\tilde{\Delta}(n)}}$$

At each n, these are "reasonable" conformal fields

$$\langle E(x)E(0)\rangle = \lim_{n\to 0} \langle E_1(x)E_1(0)\rangle = \lim_{n\to 0} \frac{1}{n}\left(\frac{A(n)}{x^{2\Delta(n)}} + (n-1)\frac{B(n)}{x^{2\tilde{\Delta}(n)}}\right) \sim \frac{\ln(x)}{x^{2\Delta(0)}}$$

Logarithms at disordered critical points are inevitable!

Conclusions

Logarithmic operators at critical points with quenched disorder are inevitable, control the structure of the CFT, and are not understood. The need to be understood if we are to develop a general theory of such critical points.

The end