Excitations in Random Elastic Media

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Collaboration with:

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Disorder in non-interacting fermionic systems

\[ H = \sum_{ij} \mathcal{H}_{ij} a_i^\dagger a_j \quad \xrightarrow{\text{Diagonalization}} \quad H = \sum_{ij} E_n \gamma_i^\dagger \gamma_j \]

\[ a_i = \sum_i \psi_i^{(n)} \gamma_n \]

\[ E_n \]

\[ E_F \]

Want to know:

- statistics of \( E_n \)
- localization properties of \( \psi^{(n)} \)
### Symmetry Classes of Random Hamiltonians

<table>
<thead>
<tr>
<th>Wigner-Dyson Classes</th>
<th>Chiral classes</th>
<th>Superconductor classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard Schrödinger equations</td>
<td>$\mathcal{H} = \begin{pmatrix} 0 &amp; Q \ Q^\dagger &amp; 0 \end{pmatrix}$ or Schrödinger equations of this type</td>
<td>$\mathcal{H} = \begin{pmatrix} h &amp; \Delta \ \Delta^\dagger &amp; -h^T \end{pmatrix}$ or Bogoliubov - de Gennes equations</td>
</tr>
<tr>
<td>$Q \psi_2 = E \psi_1$</td>
<td></td>
<td>$Q^\dagger \psi_1 = E \psi_2$</td>
</tr>
</tbody>
</table>

Disorder in bosonic systems

\[ H = \sum_{ij} \mathcal{H}_{ij} a_i^{\dagger} a_j \quad \text{Diagonalization} \quad E = \sum_{n} \omega_n \gamma_n^{\dagger} \gamma_n \]

All the frequencies must be positive. The statistics (and localization properties) of low lying levels must depend on this fact.

\[ \omega_n \]

\{ Low lying states \\
Zero energy \}
Example: interacting BEC in a random potential

\[ H = \int d^d x \left[ \frac{1}{2m} \nabla \psi^* \nabla \psi + \frac{g}{2} |\psi|^4 + (U(x) - \mu) |\psi|^2 \right] \]
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Split the condensate into the ground state and excitations

\[ \psi = \psi_0(x) + a(x, t) \]
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Random potential

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\[ -\frac{1}{2m} \Delta \psi_0 + U(x) \psi_0 + g |\psi|^2 \psi_0 = \mu \psi_0 \]
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\[
H \approx \int d^d x \left[ \frac{1}{2m} \nabla a^* \nabla a + \frac{g}{2} \left( \psi_0^* a^2 + \psi_0^2 a^* a + 4 |\psi_0|^2 a^* a \right) + (U(x) - \mu) |a|^2 \right]
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\[-i \frac{\partial a}{\partial t} = [H, a] = \frac{\delta H}{\delta a^*} \]

\[ i \frac{\partial a^*}{\partial t} = [H, a^*] = \frac{\delta H}{\delta a} \]

Equations of motion

\[ \omega a = \frac{\delta H}{\delta a^*} \]

\[ -\omega a^* = \frac{\delta H}{\delta a} \]
Generic Bosonic Harmonic Excitations

\[ H = \frac{1}{2} \sum_{ij} \left( \begin{array}{c} q_i \\ p_i \end{array} \right) \mathcal{H}_{ij} \left( \begin{array}{c} q_j \\ p_j \end{array} \right) \]

\[ \mathcal{H}_{ij} = \left( \begin{array}{cc} K_{ij} & C_{ij} \\ C_{ij}^T & M_{ij}^{-1} \end{array} \right) \]

- \( K_{ij} \) is the matrix of "spring constants"
- \( M_{ij}^{-1} \) is the matrix of "masses"
- \( C_{ij} \) is the matrix of the "Lorentz" forces
Generic Bosonic Harmonic Excitations

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\[ M_{ij}^{-1} \quad \text{is the matrix of "masses"} \]

\[ C_{ij} \quad \text{is the matrix of the "Lorentz" forces} \]

\[ a_i = q_i + ip_i \quad a_i^\dagger = q_i - ip_i \]

\[ H = \frac{1}{2} \sum_{ij} (a_i^\dagger, a_i) \mathcal{H}_{ij} \left( \begin{array}{c} a_j^\dagger \\ a_j \end{array} \right) \]

\[ h_{ij} = h_{ij}^\dagger \quad \text{is the matrix of "kinetic energy"} \]

\[ \mathcal{H}_{ij} = \begin{pmatrix} h_{ij} & \Delta_{ij} \\ \Delta_{ij}^T & h_{ij}^T \end{pmatrix} \]

\[ \Delta = \Delta^T \quad \text{is the matrix of the "anomalous pairing"} \]
Equations of motion

\[
\left( \begin{array}{c} -\frac{\partial p_i}{\partial t} \\ \frac{\partial q_i}{\partial t} \end{array} \right) = \mathcal{H}_{ij} \left( \begin{array}{c} q_j \\ p_j \end{array} \right) \quad \quad \longrightarrow \quad \quad \left( \begin{array}{c} -i\omega \ p \\ i\omega \ q \end{array} \right) = \mathcal{H}_{ij} \left( \begin{array}{c} q_j \\ p_j \end{array} \right)
\]

\[
i \left( \begin{array}{c} \frac{\partial a_i}{\partial t} \\ -\frac{\partial a_i^\dagger}{\partial t} \end{array} \right) = \mathcal{\tilde{H}}_{ij} \left( \begin{array}{c} a_j \\ a_j^\dagger \end{array} \right) \quad \quad \longrightarrow \quad \quad \left( \begin{array}{c} \omega a_i \\ -\omega a_i^\dagger \end{array} \right) = \mathcal{\tilde{H}}_{ij} \left( \begin{array}{c} a_j \\ a_j^\dagger \end{array} \right)
\]

Bottom line: must solve \[\omega \Sigma \psi = \mathcal{H} \psi \quad \quad \Sigma^2 = 1\]

\[\mathcal{H}\] must be positive definite: it follows that the frequencies are real and come in opposite pairs \(\pm \omega\)
Types of bosonic excitations

1. \( \mathcal{H} = \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix} \)

\[ \mathcal{H} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \quad \tilde{\mathcal{H}} = \begin{pmatrix} h & 0 \\ 0 & h^T \end{pmatrix} \]

\[ h = h^* \]

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2. \[ \mathcal{H} = \begin{pmatrix} K & 0 \\ 0 & M^{-1} \end{pmatrix} \quad \tilde{\mathcal{H}} = \begin{pmatrix} h & \Delta \\ \Delta & h^T \end{pmatrix} \quad h = h^* \quad \Delta = \Delta^* \quad \omega^2 = \text{eigenvalues } (KM^{-1}) \]
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\[ h = h^* \]
\[ \Delta = \Delta^* \]

3. Arbitrary \( \mathcal{H} \quad \implies \quad \omega = \text{eigenvalues} (\Sigma \mathcal{H}) \)
The classification of the low frequency modes

\[ \mathcal{H} = \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix} \]

\[ \mathcal{H} = \begin{pmatrix} K & 0 \\ 0 & M^{-1} \end{pmatrix} \]

**Goldstone modes**

- "Phonons"
  - Magnons in collinear random magnets
  - Magnons in arbitrary random magnets

**Other low freq modes**

- Pinned charge density waves, randomly pinned elastic media

Phonons in 1D

\[ H = \int dx \left[ \frac{J(x)}{2} (\nabla u)^2 + \frac{p^2}{2\rho(x)} \right] \]

Random spring constants

\[ \omega^2 u = -\frac{1}{\rho(x)} \nabla [J(x)\nabla] u \]

Random masses

S. Alexander, J. Bernasconi, W. Schneider, RMP (1981)
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No disorder: \( \omega^2 \sim k^2 \)
\( \rho(\omega) \equiv \sum_k \delta(\omega - \omega_k) \sim \text{const} \)
Phonons in 1D

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Random masses

No disorder: $\omega^2 \sim k^2$ $\rho(\omega) \equiv \sum_k \delta(\omega - \omega_k) \sim \text{const}$

Presence of disorder: disorder gets progressively weaker as the wavelength $1/k$ goes to 0. The low frequency density of states is unaffected.

$$J(x) = J_0 + \delta J(x) \quad \rightarrow \quad \sqrt{\int_{1/k} dx \delta J^2} \quad \sqrt{\frac{\int_{1/k} dx J_0}{\int_{1/k} dx J_0}} \sim \sqrt{k} \rightarrow 0$$

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However, the low frequency modes develop finite lifetime \( \tau^{-1} \sim \omega^2 \).

\[ l = \sqrt{\frac{J_0}{\rho_0}} \tau \sim \omega^{-2} \quad \text{Mean free path} \]

Thus the modes are localized with the localization length

\[ \ell(\omega) \sim \frac{1}{\omega^2} \]

S. Alexander, J. Bernasconi, W Schneider, RMP (1981)
Phonons in 2D and 3D

The DoS argument still applies: \( \rho(\omega) \sim \omega^{d-1} \)
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Theory of Anderson localization in 2D: the localization length is exponentially large in \( k_F l \).

\[ k_F \sim k \]
\[ l \sim \tau \sim 1/\omega^3 \]
\[ \ell \sim \exp(k_F l) \sim \exp(1/\omega^2) \]
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\[
\begin{align*}
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    l & \sim \tau \sim 1/\omega^3 \\
    \ell & \sim \exp(k_F l) \sim \exp\left(1/\omega^2\right)
\end{align*}
\]

Theory of Anderson transitions in 3D: phonons delocalize at \( \omega \leq \omega_c \)

“The Boson Peak”

$$\omega^2 u = -\frac{1}{\rho(x)} \nabla [J(x) \nabla] u$$

Low frequency phonons average disorder over their wavelength

$$\rho(\omega) \sim \int d^{d}k \delta \left( \omega - \langle \frac{p}{J} \rangle^{-\frac{1}{2}} k \right) \sim \omega^{d-1} \langle \frac{p}{J} \rangle^{\frac{d}{2}}$$

High frequency phonons adjust to disorder at different regions of space

$$\frac{\rho(r)}{J(r)} \omega^2 = k^2$$

$$\rho(\omega) \sim \omega^{d-1} \left\langle \left( \frac{\rho(r)}{J(r)} \right)^{\frac{d}{2}} \right\rangle$$

VG, A. Altland, PRL (2005)
Magnons

\[ H = -\sum_i J_i \vec{S}_i \vec{S}_{i+1} \]

Holstein-Primakoff bosons

\[ S^+ \sim a^\dagger \]

\[ S^- \sim a \]

Equations of motion are in the "magnon" class

\[ \tilde{\mathcal{H}} = \begin{pmatrix} h & \Delta \\ \Delta & h^T \end{pmatrix} \]

\[ \tilde{\mathcal{H}} = \tilde{\mathcal{H}}^* \]

Equations of motion can be reduced to

\[ -\Delta u = \omega J(x) u \]

D. Sherrington, J. Phys. C (1979)
Magnons in 1D

\[-\Delta u = \omega J(x) u\]

Averaging over the wavelength $1/k$ \[J(x) \to \sqrt{\langle J^2(x) \rangle} \to k^{\frac{1}{2}}\]

\[k^2 \sim k^{\frac{1}{2}} \omega\]

\[\omega \sim k^{\frac{3}{2}}\]

Density of states \[\rho(\omega) \sim \int dk \delta \left( \omega - k^{\frac{1}{3}} \right) \sim \frac{1}{\omega^{\frac{1}{3}}}\]

Localization length \[\ell(\omega) \sim \frac{1}{k} \sim \frac{1}{\omega^{\frac{2}{3}}}\]

Magnons in 2D and 3D

$$-\Delta u = \omega J(x) u$$

2D: critical dimensionality

$$\rho(\omega) \sim \omega |\log(\omega)|$$

$$\ell(\omega) \sim \omega^{-\frac{1}{16\pi}}$$

3D: above the critical dimensionality

$$\rho(\omega) \sim \omega^2$$

No localization for $\omega < \omega_c$.

What if there are no Goldstone modes

Random field magnet:

Equilibrium position

\[ \phi \approx \phi_0 + \psi \]

oscillations about the equilibrium

\[ -\Delta \phi_0 + \left. \frac{\partial h}{\partial \phi} \right|_{\phi=\phi_0} = 0 \]

equilibrium equation

\[ -\Delta \left. + \frac{\partial^2 h}{\partial \phi^2} \right|_{\phi=\phi_0} \psi = \omega^2 \psi \]

oscillation equation

\[ H = \int dx \left[ \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + h(\phi, x) \right] - h_i \cos(\phi_i - \chi_i) \]

Despite superficial similarity, this is not a random Schrödinger equation.

H. Fukuyama, P. Lee, PRB (1978)
Mapping to a Chiral Random Problem

Taking the "square root"

\[
\left[-\Delta + \frac{\partial^2 h}{\partial \phi^2}\right]_{\phi=\phi_0} \psi = \omega^2 \psi \quad \rightarrow \quad \left[\frac{d}{dx} + V(x)\right] \left[-\frac{d}{dx} + V(x)\right] \psi = \omega^2 \psi
\]

\[
\begin{pmatrix}
0 & \frac{d}{dx} + V(x) \\
-\frac{d}{dx} + V(x) & 0
\end{pmatrix}
\begin{pmatrix}
\psi \\
\tilde{\psi}
\end{pmatrix} = \omega
\begin{pmatrix}
\psi \\
\tilde{\psi}
\end{pmatrix}
\]

Density of states

\[
\rho(\omega) \sim \omega^4 \\
\omega \lesssim \omega_c
\]

\[
\ell(\omega) \sim \omega
\]

I. Aleiner, I. Ruzin, PRL (1993)
Back to disordered BEC, in 1D

$$H = \int dx \left[ \frac{1}{2} \nabla \psi^* \nabla \psi + \frac{g}{2} |\psi|^4 + (U(x) - \mu) |\psi|^2 \right]$$

$$\psi = \rho e^{i\phi}$$  
Having found $\psi_0 = \rho_0(x)$, look for fluctuations $\delta \rho$, $\delta \phi$
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This is the fluctuation equation

\[ \frac{1}{\rho_0} \left[ -\frac{d^2}{dx^2} + U(x) - \mu + 3 \rho_0^2 \right] \frac{1}{\rho_0} \left[ -\frac{d}{dx} \rho_0 \frac{d}{dx} \right] \delta \phi = \omega^2 \delta \phi \]
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Having found \( \psi_0 = \rho_0(x) \), look for fluctuations \( \delta \rho, \delta \phi \)

This is the fluctuation equation

\[ \rho_0 = \text{const}, \quad \left[ k^2 + \text{const} \right] k^2 = \omega^2 \]
Back to disordered BEC, in 1D

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Formally, looks like "magnons"...

This piece is phonons!
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Localization length \( \ell \sim \frac{1}{\omega^2} \)

\( \ell \sim \omega \)

VG, unpublished (2007)
Summary

- Phonons: $\rho(\omega) \sim \omega^{d-1}$
  
  \begin{align*}
    1D &: \quad l(\omega) \sim \frac{1}{\omega^2} \\
    2D &: \quad l(\omega) \sim \exp\left(\frac{1}{\omega^2}\right) \\
    3D &: \quad \text{delocalization}
  \end{align*}

- Magnons:
  
  \begin{align*}
    1D &: \quad \rho(\omega) \sim \frac{1}{\omega^{\frac{1}{3}}} \\
    2D &: \quad \rho(\omega) \sim \omega |\log \omega| \\
    3D &: \quad \rho(\omega) \sim \omega^2 \\
    1D &: \quad \ell(\omega) \sim \frac{1}{\omega^{\frac{2}{3}}} \\
    2D &: \quad \ell(\omega) \sim \frac{1}{\omega^{\frac{1}{16\pi}}} \\
    3D &: \quad \text{delocalization}
  \end{align*}

- Pinned elastic media: $1D : \quad \rho(\omega) \sim \omega^4$, $\ell(\omega) \sim \text{const}$

- Open questions:
  a) Are "magnons" universal?
  b) Other classes of behavior?
  c) Pinning at $D>1$?
  d) Experiment?