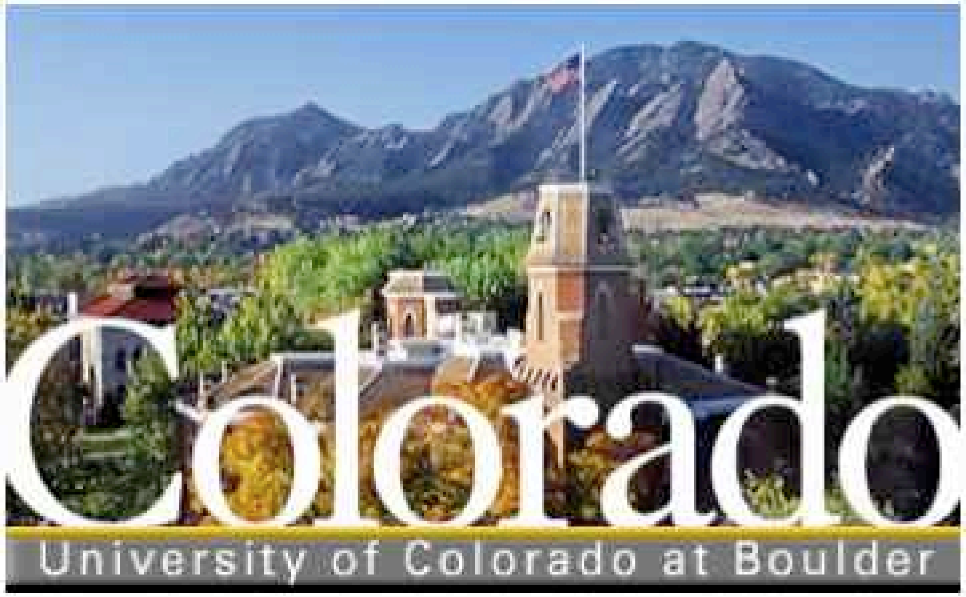


Excitations in Random Elastic Media

Victor Gurarie

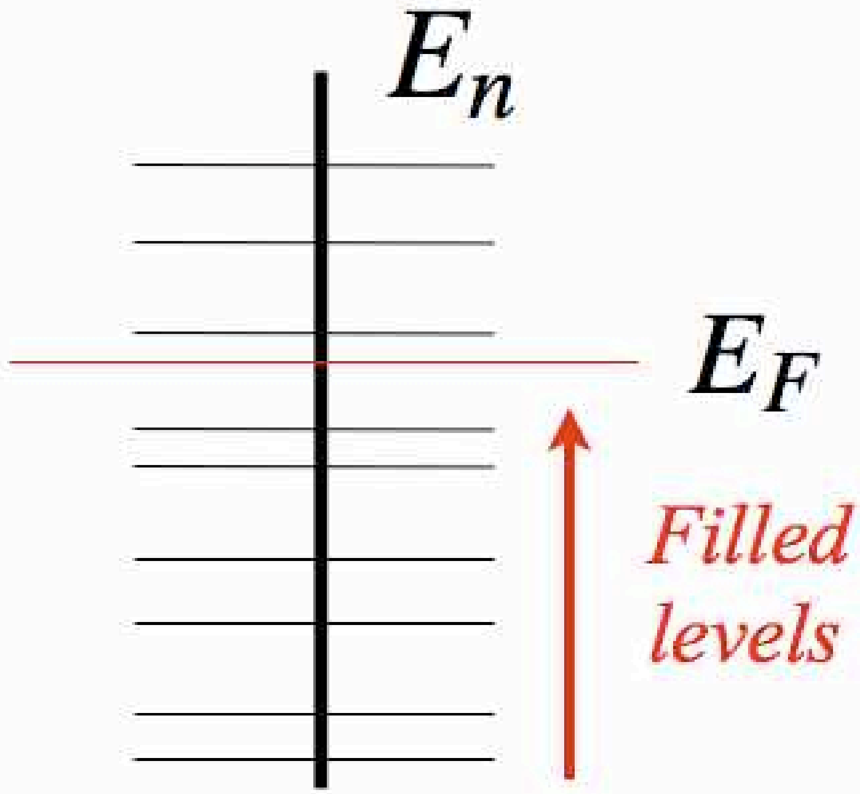
Collaboration with:

- J. Chalker (Oxford)
- A. Altland (Cologne)

Disorder in non-interacting fermionic systems

$$H = \sum_{ij} \mathcal{H}_{ij} a_i^\dagger a_j \xrightarrow{\text{Diagonalization}} H = \sum_{ij} E_n \gamma_n^\dagger \gamma_n$$

$$a_i = \sum_i \psi_i^{(n)} \gamma_n$$



Want to know:

- statistics of E_n
- localization properties of $\psi^{(n)}$

Symmetry Classes of Random Hamiltonians

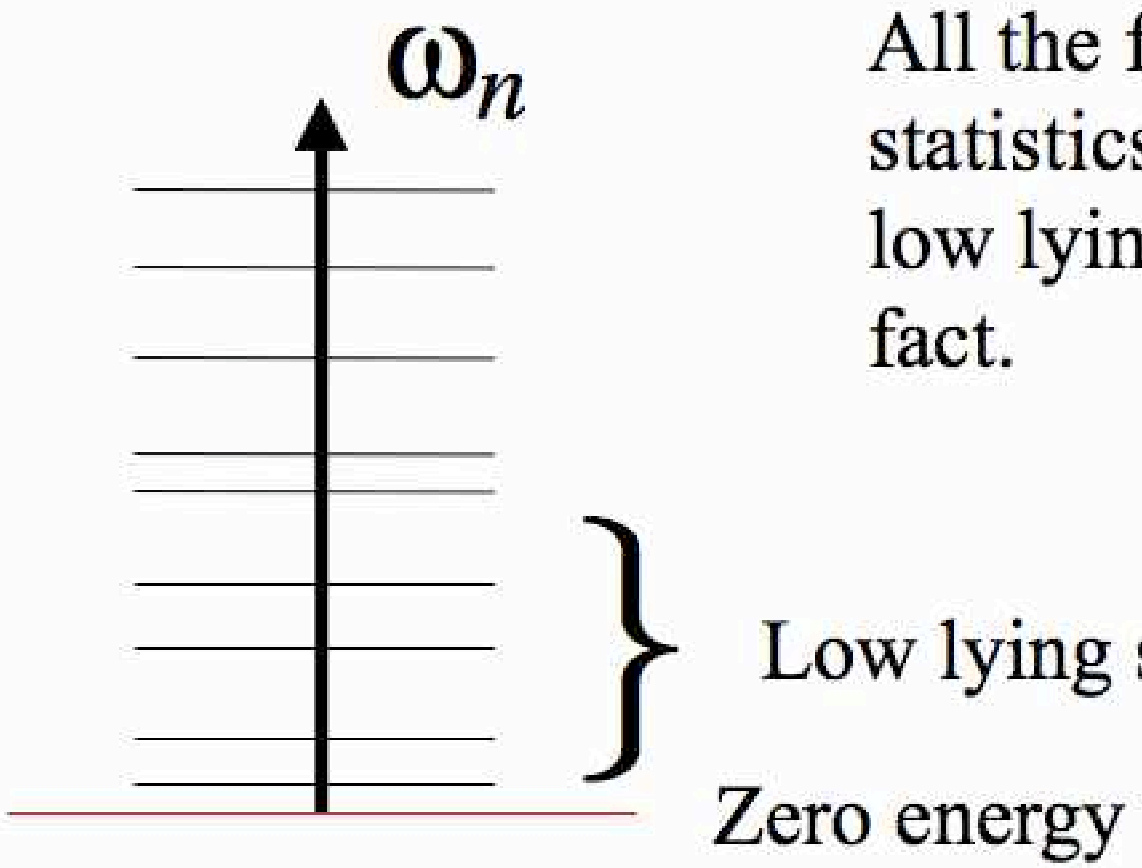
Wigner-Dyson Classes	Chiral classes	Superconductor classes
Standard Schrödinger equations	$\mathcal{H} = \begin{pmatrix} 0 & Q \\ Q^\dagger & 0 \end{pmatrix}$ or Schrödinger equations of this type $Q\psi_2 = E\psi_1$ $Q^\dagger\psi_1 = E\psi_2$	$\mathcal{H} = \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^T \end{pmatrix}$ or Bogoliubov - de Gennes equations

M. Zirnbauer, J. Math. Phys. (1996)

Disorder in bosonic systems

Diagonalization

$$H = \sum_{ij} \mathcal{H}_{ij} a_i^\dagger a_j \longrightarrow E = \sum_n \omega_n \gamma_n^\dagger \gamma_n$$



All the frequencies must be positive. The statistics (and localization properties) of low lying levels must depend on this fact.

Example: interacting BEC in a random potential

$$H=\int d^d x\left[\frac{1}{2m}\nabla\psi^*\nabla\psi+\frac{g}{2}|\psi|^4+(U(x)-\mu)|\psi|^2\right]$$

Random potential

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Random potential

Split the condensate into the ground state and excitations $\psi = \psi_0(x) + a(x,t)$

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$$H \approx \int d^d x \left[\frac{1}{2m} \nabla a^* \nabla a + \frac{g}{2} \left(\psi_0^{*2} a^2 + \psi_0^2 a^{*2} + 4 |\psi_0|^2 a^* a \right) + (U(x) - \mu) |a|^2 \right]$$

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$$-i\frac{\partial a}{\partial t} = [H, a] = \frac{\delta H}{\delta a^*}$$

$$i\frac{\partial a^*}{\partial t} = [H, a^*] = \frac{\delta H}{\delta a}$$

Equations of motion

$$\omega\, a = \frac{\delta H}{\delta a^*}$$

$$-\omega\, a^* = \frac{\delta H}{\delta a}$$

Generic Bosonic Harmonic Excitations

$$H = \frac{1}{2}\sum_{ij} \begin{pmatrix} q_i & p_i \end{pmatrix} \mathcal{H}_{ij} \begin{pmatrix} q_j \\ p_j \end{pmatrix}$$

$$\mathcal{H}_{ij} = \begin{pmatrix} K_{ij} & C_{ij} \\ C_{ij}^T & M_{ij}^{-1} \end{pmatrix}$$

K_{ij} is the matrix of "spring constants"

M_{ij}^{-1} is the matrix of "masses"

C_{ij} is the matrix of the "Lorentz" forces

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$$a_i = q_i + ip_i \qquad a_i^\dagger = q_i - ip_i$$

$$H = \frac{1}{2}\sum_{ij} \begin{pmatrix} a_i^\dagger & a_i \end{pmatrix} \tilde{\mathcal{H}}_{ij} \begin{pmatrix} a_j \\ a_j^\dagger \end{pmatrix}$$

$$\tilde{\mathcal{H}}_{ij} = \begin{pmatrix} h_{ij} & \Delta_{ij} \\ \Delta_{ij}^\dagger & h_{ij}^T \end{pmatrix}$$

$h_{ij} = h_{ij}^\dagger$ is the matrix of "kinetic energy"

$\Delta = \Delta^T$ is the matrix of the "anomalous pairing"

Equations of motion

$$\begin{pmatrix} -\frac{\partial p_i}{\partial t} \\ \frac{\partial q_i}{\partial t} \end{pmatrix} = \mathcal{H}_{ij} \begin{pmatrix} q_j \\ p_j \end{pmatrix} \longrightarrow \begin{pmatrix} -i\omega\, p \\ i\omega\, q \end{pmatrix} = \mathcal{H}_{ij} \begin{pmatrix} q_j \\ p_j \end{pmatrix}$$

$$i \begin{pmatrix} \frac{\partial a_i}{\partial t} \\ -\frac{\partial a_i^\dagger}{\partial t} \end{pmatrix} = \tilde{\mathcal{H}}_{ij} \begin{pmatrix} a_j \\ a_j^\dagger \end{pmatrix} \longrightarrow \begin{pmatrix} \omega a_i \\ -\omega a_i^\dagger \end{pmatrix} = \tilde{\mathcal{H}}_{ij} \begin{pmatrix} a_j \\ a_j^\dagger \end{pmatrix}$$

Bottom line: must solve $\omega \Sigma \psi = \mathcal{H} \psi$ $\Sigma^2 = 1$

$\mathcal{H}$ must be positive definite: it follows that the frequencies are real and come in opposite pairs $\pm\omega$

Types of bosonic excitations

1. $\mathcal{H} = \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix}$ $\longrightarrow$ $\omega^2 = \text{ eigenvalues } (K)$

$\mathcal{H} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$ $\tilde{\mathcal{H}} = \begin{pmatrix} h & 0 \\ 0 & h^T \end{pmatrix}$ $\longrightarrow$ $\omega = |\text{eigenvalues } (K)|$

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2. $\mathcal{H} = \begin{pmatrix} K & 0 \\ 0 & M^{-1} \end{pmatrix}$ $\tilde{\mathcal{H}} = \begin{pmatrix} h & \Delta \\ \Delta & h^T \end{pmatrix}$ $\longrightarrow$ $\omega^2 = \text{ eigenvalues } (KM^{-1})$

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$$h = h^*$$
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3. Arbitrary $\mathcal{H}$ $\longrightarrow$ $\omega = \text{ eigenvalues } (\Sigma\mathcal{H})$

The classification of the low frequency modes

	Goldstone modes	Other low freq modes
$\mathcal{H} = \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix}$	"Phonons"	Pinned charge density waves, randomly pinned elastic media
$\mathcal{H} = \begin{pmatrix} K & 0 \\ 0 & M^{-1} \end{pmatrix}$	Magnons in collinear random magnets	
Arbitrary $\mathcal{H}$	Magnons in arbitrary random magnets	

VG, J. Chalker, PRB (2003)

Phonons in 1D

Random spring constants

$$H = \int dx \left[\frac{J(x)}{2} (\nabla u)^2 + \frac{p^2}{2\rho(x)} \right] \longrightarrow \omega^2 u = -\frac{1}{\rho(x)} \nabla \left[J(x) \nabla \right] u$$

Random masses

S. Alexander, J. Bernasconi, W Schneider, RMP (1981)

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No disorder: $\omega^2 \sim k^2 \quad \rho(\omega) \equiv \sum_k \delta(\omega - \omega_k) \sim$ const

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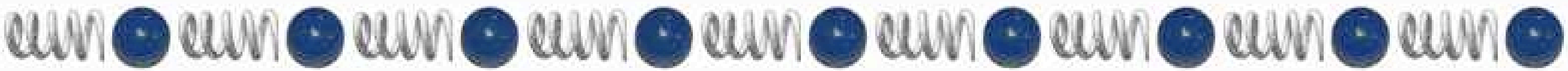

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Presence of disorder: disorder gets progressively weaker as the wavelength *1/k* goes to 0.
The low frequency density of states is unaffected.

$$J(x) = J_0 + \delta J(x) \longrightarrow \frac{\sqrt{\int_{1/k} dx\, \delta J^2}}{\int_{1/k} dx\, J_0} \sim \sqrt{k} \to 0$$

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However, the low frequency modes develop finite lifetime $\tau^{-1} \sim \omega^2$. $\quad l = \sqrt{\frac{J_0}{\rho_0}} \tau \sim \omega^{-2}$ Mean free path

Thus the modes are localized with the localization length $\ell(\omega) \sim \frac{1}{\omega^2}$

S. Alexander, J. Bernasconi, W Schneider, RMP (1981)

Phonons in 2D and 3D

The DoS argument still applies: $\rho(\omega) \sim \omega^{d-1}$

2D, 3D: S. John, H. Sompolinsky, M Stephen, PRB (1983).

Phonons in 2D and 3D

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Theory of Anderson localization in 2D: the localization length is exponentially large in $k_F\, l$.

Mean free path

$k_F \sim k$

$l \sim \tau \sim 1/\omega^3$

$$\ell \sim \exp(k_F l) \sim \exp\left(1/\omega^2\right)$$

localization length

2D, 3D: S. John, H. Sompolinsky, M Stephen, PRB (1983).

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localization length

Theory of Anderson transitions in 3D: phonons delocalize at

$\omega \leq \omega_c$

2D, 3D: S. John, H. Sompolinsky, M Stephen, PRB (1983).

"The Boson Peak"

$$\omega^2 u = -\frac{1}{\rho(x)} \nabla \left[J(x) \nabla \right] u$$

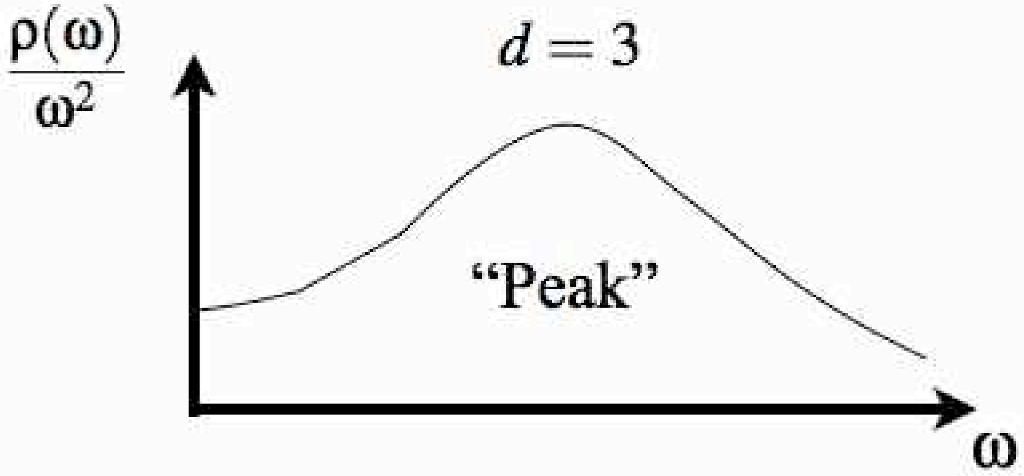
Low frequency phonons average disorder over their wavelength

$$\left\langle \frac{\rho}{J} \right\rangle \omega^2 = k^2$$

$$\rho(\omega) \sim \int d^d k \, \delta\left(\omega - \left\langle \frac{\rho}{J} \right\rangle^{-\frac{1}{2}} k \right) \sim \omega^{d-1} \left\langle \frac{\rho}{J} \right\rangle^{\frac{d}{2}}$$

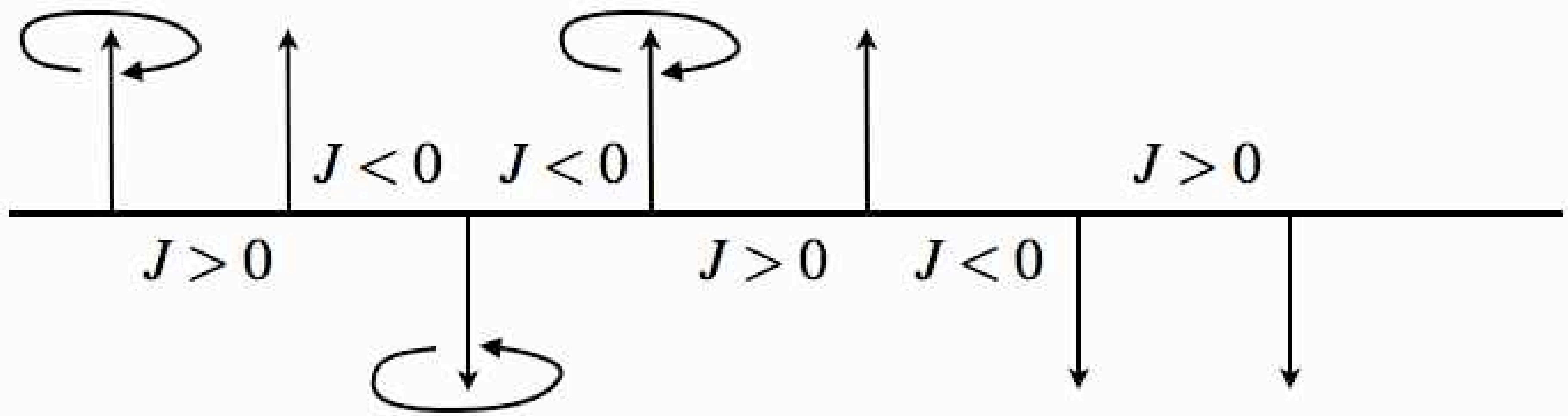
High frequency phonons adjust to disorder at different regions of space

$$\frac{\rho(r)}{J(r)} \omega^2 = k^2 \qquad \rho(\omega) \sim \omega^{d-1} \left\langle \left(\frac{\rho(r)}{J(r)} \right)^{\frac{d}{2}} \right\rangle$$



J. Chalker, unpublished (2003)
VG, A. Altland, PRL (2005)

Magnons



$H = -\sum_i J_i \vec{S}_i \vec{S}_{i+1}$

Holstein-Primakoff bosons $S^+ \sim a^\dagger$, $S^- \sim a$

Equations of motion are in the "magnon" class $\tilde{\mathcal{H}} = \begin{pmatrix} h & \Delta \\ \Delta & h^T \end{pmatrix}$ $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}^*$

Equations of motion can be reduced to $-\Delta u = \omega J(x)\, u$

D. Sherrington, J. Phys. C (1979)

Magnons in 1D

$$-\Delta u = \omega J(x)\, u$$

Averaging over the wavelength *1/k* $\quad J(x) \to \sqrt{\langle J^2(x) \rangle} \to k^{\frac{1}{2}}$

$$k^2 \sim k^{\frac{1}{2}} \omega$$

$$\omega \sim k^{\frac{3}{2}}$$

Density of states $\quad \rho(\omega) \sim \int dk\, \delta\left(\omega - k^{\frac{1}{3}}\right) \sim \frac{1}{\omega^{\frac{1}{3}}}$

Localization length $\quad \ell(\omega) \sim \frac{1}{k} \sim \frac{1}{\omega^{\frac{2}{3}}}$

R. B. Stinchcombe, I.R. Pimentel, PRB (1988)

Magnons in 2D and 3D

$$-\Delta u = \omega J(x)\, u$$

2D: critical dimensionality

$$\rho(\omega) \sim \omega \left|\log(\omega)\right|$$
$$\ell(\omega) \sim \omega^{-\frac{1}{16\pi}}$$

3D: above the critical dimensionality

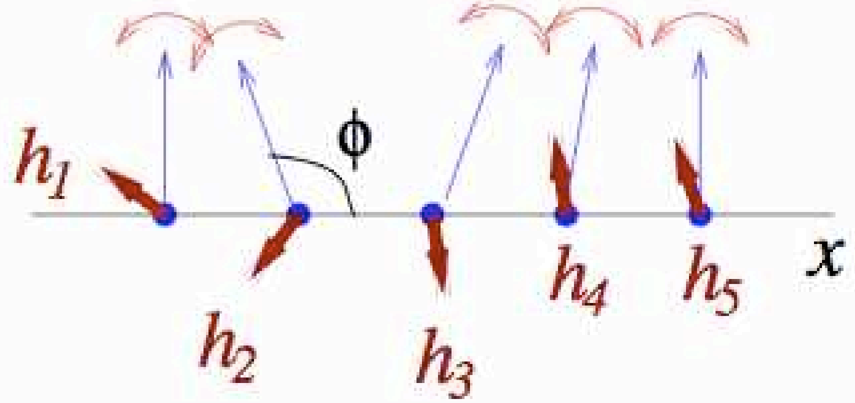
$$\rho(\omega) \sim \omega^2$$

No localization for $\omega < \omega_c$.

R. Johnston, D. Sherrington, J. Phys. C (1982)
VG, A. Altland, J. Phys. A (2004)

What if there are no Goldstone modes

Random field magnet:



$$H = \int dx \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + h(\phi, x) \right].$$

$h(\phi, x)$: $-h_i \cos(\phi_i - \chi_i)$

Equlibrium position; oscillations about the equilibrium

$$\phi \approx \phi_0 + \psi$$

$$-\Delta\phi_0 + \left.\frac{\partial h}{\partial \phi}\right|_{\phi=\phi_0} = 0$$ equilibrium equation

$$\left[-\Delta + \left.\frac{\partial^2 h}{\partial \phi^2}\right|_{\phi=\phi_0} \right] \psi = \omega^2 \psi$$ oscillation equation

Despite superficial similarity, this is not a random Schrödinger equation.

H. Fukuyama, P. Lee, PRB (1978)

Mapping to a Chiral Random Problem

Taking the "square root"

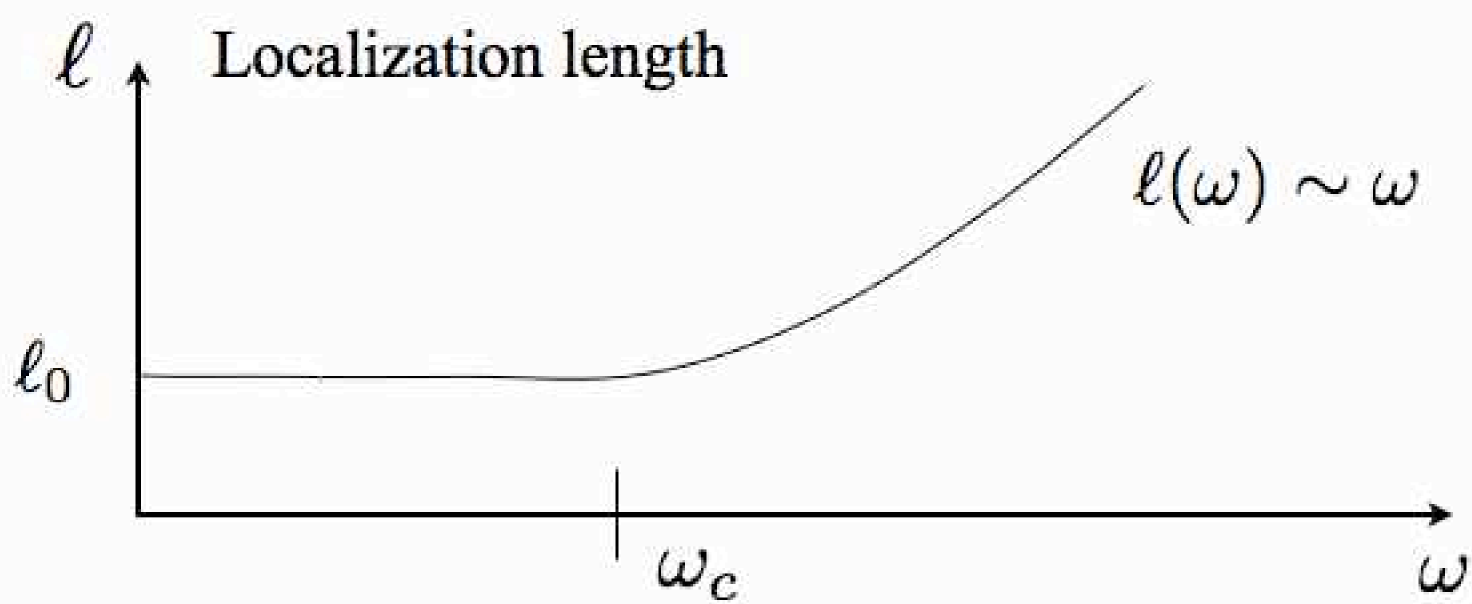
$$\left[-\Delta + \frac{\partial^2 h}{\partial \phi^2}\bigg|_{\phi=\phi_0}\right]\psi = \omega^2\psi \longrightarrow \left[\frac{d}{dx} + V(x)\right]\left[-\frac{d}{dx} + V(x)\right]\psi = \omega^2\psi$$

$$\begin{pmatrix} 0 & \frac{d}{dx} + V(x) \\ -\frac{d}{dx} + V(x) & 0 \end{pmatrix}\begin{pmatrix}\psi \\ \tilde{\psi}\end{pmatrix} = \omega\begin{pmatrix}\psi \\ \tilde{\psi}\end{pmatrix}$$

Density of states

$\rho(\omega) \sim \omega^4$

$\omega \lesssim \omega_c$



I. Aleiner, I. Ruzin, PRL (1993)
VG, J. Chalker, PRL (2002)

Back to disordered BEC, in 1D

$$H=\int dx \left[\frac{1}{2}\nabla\psi^*\nabla\psi + \frac{g}{2}\left|\psi\right|^4 + (U(x)-\mu)\left|\psi\right|^2\right]$$

$\psi = \rho e^{i\phi}$ Having found $\psi_0 = \rho_0(x)$, look for fluctuations $\delta\rho,\ \delta\phi$

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This is the fluctuation equation

$$\frac{1}{\rho_0}\left[-\frac{d^2}{dx^2}+U(x)-\mu+3\rho_0^2\right]\frac{1}{\rho_0}\left[-\frac{d}{dx}\rho_0^2\frac{d}{dx}\right]\delta\phi = \omega^2\delta\phi$$

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$$\rho_0 = \text{const},\ \left[k^2 + \text{const}\right] k^2 = \omega^2$$

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Formally, looks like "magnons"...

This piece is phonons!

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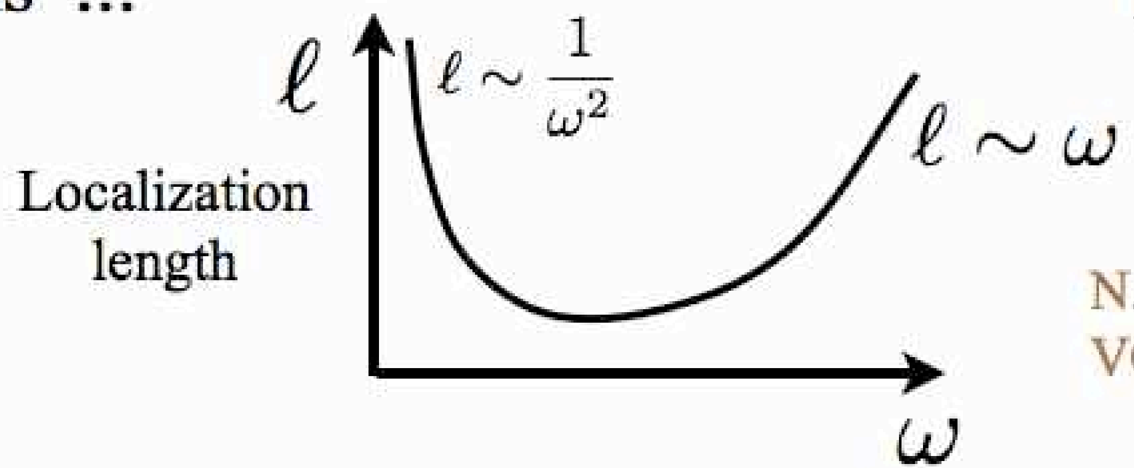

N. Bilas, N. Pavloff, Eur. Phys. J. D (2006)
VG, unpublished (2007)

Summary

- Phonons: $\rho(\omega) \sim \omega^{d-1}$

  $1D: \; l(\omega) \sim 1/\omega^2$
  $2D: \; l(\omega) \sim \exp\left(1/\omega^2\right)$
  $3D: \;$ delocalization

- Magnons:

  $1D: \; \rho(\omega) \sim 1/\omega^{\frac{1}{3}}$ $\quad$ $1D: \; \ell(\omega) \sim 1/\omega^{\frac{2}{3}}$
  $2D: \; \rho(\omega) \sim \omega\, |\log \omega|$ $\quad$ $2D: \; \ell(\omega) \sim 1/\omega^{\frac{1}{16\pi}}$
  $3D: \; \rho(\omega) \sim \omega^2$ $\quad$ $3D: \;$ delocalization

- Pinned elastic media: $1D: \; \rho(\omega) \sim \omega^4, \; \ell(\omega) \sim$ const

- Open questions: a) Are "magnons" universal?
  b) Other classes of behavior?
  c) Pinning at D>1?
  d) Experiment?