LECTURE 3: CURVES IN EUCLIDEAN AND AFFINE SPACES

1. INTRODUCTION

When studying submanifolds Σ of a homogenous space G/H, one question that arises frequently is the notion of *equivalence:* given two submanifolds Σ_1, Σ_2, when can one be transformed into the other via a symmetry of the underlying space G/H? For instance, in the case of submanifolds of Euclidean space $\mathbb{E}^n$ we say that two submanifolds are equivalent if one can be transformed into the other by a rigid motion.

We will approach this general question by considering the restriction of certain frames on the underlying space G/H to the submanifold Σ in question. If $f : \Sigma \to G/H$ is an immersion, then "restriction" actually means pullback: the restriction of a frame to Σ is really a section of the bundle $f^{-1}(G)$ lying over Σ. We will find it useful to consider frames that are *adapted* to Σ. This means that instead of considering arbitrary frames, we will use the geometry of Σ to choose "nice" frames. This is somewhat analogous to choosing "nice" coordinates at a point on a surface to study the geometry at that point. The beauty of the method of moving frames is that we can do this at all points simultaneously.

Once we have chosen a nice frame along Σ, we can consider the restriction of the Maurer-Cartan form ω and its structure equations to this frame. Again, by "restriction" we actually mean the pullback $f^*\omega$. The structure equations will generally contain quantities which are *invariants* of Σ; this means that if we change Σ by a symmetry of the ambient space G/H, these quantities remain unchanged. Typical examples of invariants are quantities such as arc length, curvature, etc.

Given an immersion $f : \Sigma \to G/H$, the choice of a frame amounts to a lifting of f to a map $\tilde{f} : \Sigma \to G$. In order for the frame to contain useful information about the invariants of Σ, the choice of this lifting should be completely determined in some canonical way by the geometry of Σ. Moreover, the lifting itself should be invariant in the sense that

$$\widetilde{(g \cdot f)} = g \cdot \tilde{f}$$

for any $g \in G$. If such an invariant lifting exists, then the question of equivalence is completely answered by the following lemma:

Important Lemma 1: Let $\tilde{f}_1, \tilde{f}_2 : \Sigma \to G$ be two smooth immersions. Then there exists an element $g \in G$ such that

$$\tilde{f}_1(x) = g\tilde{f}_2(x)$$

for all $x \in G$ if and only if $\tilde{f}_1^*\omega = \tilde{f}_2^*\omega$, where ω is the Maurer-Cartan form of G.

This is not too surprising; if the restriction $\tilde{f}^*\omega$ contains information about the invariants of Σ, then this information should remain the same if Σ is transformed by an element of G. The converse says that, in fact, $\tilde{f}^*\omega$ contains enough information about Σ to determine it completely up to a symmetry of the ambient space.

2. CURVES IN $\mathbb{E}^3$

Consider a smooth parametrized curve $x : I \to \mathbb{E}^3$ which maps some open interval $I \subset \mathbb{R}$ into Euclidean space. A frame along x is a choice, for each $t \in I$, of an orthonormal frame $\{e_1(t), e_2(t), e_3(t)\}$ for the tangent space $T_{x(t)}\mathbb{E}^3$. If the curve is "nice enough" (the precise meaning of this will become clear shortly), we can choose such a frame in a way that reflects the geometry of the curve.

Recall that x is *regular* if $x'(t) \neq 0$ for every $t \in I$. The first condition we will require in order that x be "nice enough" is that x must be a regular curve. With this assumption, we can make our first frame adaptation by setting

$$e_1(t) = \frac{x'(t)}{|x'(t)|},$$

i.e., we require that $e_1(t)$ be the unit tangent vector to the curve at $x(t)$. This choice is clearly invariant under the action of $E(3)$. The vector e_1 is now uniquely determined, but we may still vary e_2, e_3 by an arbitrary rotation in $O(2)$.

Here we make an observation that will simplify the remainder of our computations. Fix $t_0 \in I$ and define the *arc length* of x at $t \in I$ to be

$$s(t) = \int_{t_0}^{t} |x'(\tau)|\, d\tau.$$

The arc length is clearly invariant under the action of $E(3)$, and since $x'(t) \neq 0$ for all $t \in I$, $s(t)$ has an inverse function $t(s)$. By setting $x(s) = x(t(s))$, we can assume that x is parametrized by arc length, and so $e_1(s) = x'(s)$.

In order to make the next adaptation, we need to make another assumption about the curve. We will say that x is *nondegenerate* if x is regular and, in addition, $e_1'(s) \neq 0$ for all $s \in I$. In this case, differentiating the equation

$$\langle e_1(s), e_1(s) \rangle = 1$$

with respect to s yields

$$\langle e_1'(s), e_1(s)\rangle = 0.$$

Thus $e_1'(s)$ is orthogonal to $e_1(s)$, and we can make our second adaptation by setting

$$e_2(s) = \frac{e_1'(s)}{|\, e_1'(s)\,|}.$$

This vector is called the *unit normal vector* to the curve at $x(s)$. It has the property that the *osculating plane* to the curve at $x(s)$ is spanned by $e_1(s)$ and $e_2(s)$. This choice is also invariant under the action of $E(3)$.

The frame is now essentially unique; because the frame must be orthonormal, $e_3(s)$ is determined up to multiplication by ± 1. (Indeed, if we consider *oriented* orthonormal frames, $e_3(s)$ is uniquely determined.) $e_3(s)$ is called the *binormal vector* to the curve at $x(s)$. This adapted frame $\{e_1(s), e_2(s), e_3(s)\}$ is called the *Frenet frame* of the curve $x(s)$ and defines a lifting of the curve $\tilde{x} : I \to E(3)$.

Now we want to consider the Maurer-Cartan forms of $E(3)$ *restricted to the Frenet frame.* When we pull these forms back via the map $\tilde{x} : I \to E(3)$, they become forms on the 1-dimensional space I, and so they must all be multiples of ds. For instance, $\tilde{x}^*(dx) = x'(s)\, ds$. Recall that the structure equations on $E(3)$ are

$$dx = \sum_{i=1}^{3} e_i\, \omega^i$$

$$de_i = \sum_{j=1}^{3} e_j\, \omega_i^j$$

and that $\omega_j^i = -\omega_i^j$. Let $\frac{\partial}{\partial s}$ denote the vector field on I which satisfies $ds(\frac{\partial}{\partial s}) = 1$. Pulling back the structure equations via $\tilde{x}$ and evaluating all the 1-forms on the vector field $\frac{\partial}{\partial s}$ yields

$$x'(s) = \sum_{i=1}^{3} e_i(s)\, \omega^i(\tfrac{\partial}{\partial s})$$

$$e_i'(s) = \sum_{j=1}^{3} e_j(s)\, \omega_i^j(\tfrac{\partial}{\partial s}).$$

But we have chosen our frame so that $x'(s) = e_1(s)$ and $e_1'(s)$ is a multiple of $e_2(s)$, say $e_1'(s) = \kappa(s)\, e_2(s)$. The function $\kappa(s)$ is called the *curvature* of x at s. Note that x is nondegenerate if and only if $\kappa(s) \neq 0$ for all $s \in I$.

The structure equations imply that

$$x'(s) = e_1(s) = \sum_{i=1}^{3} e_i(s)\,\omega^i(\tfrac{\partial}{\partial s})$$

$$e_1'(s) = \kappa(s)\,e_2(s) = \sum_{j=1}^{3} e_j(s)\,\omega_1^j(\tfrac{\partial}{\partial s}).$$

Therefore,

$$\omega^1(\tfrac{\partial}{\partial s}) = 1, \qquad \omega^2(\tfrac{\partial}{\partial s}) = 0, \qquad \omega^3(\tfrac{\partial}{\partial s}) = 0,$$

$$\omega_1^2(\tfrac{\partial}{\partial s}) = \kappa(s), \qquad \omega_1^3(\tfrac{\partial}{\partial s}) = 0.$$

Now define the function $\tau(s)$ by

$$\tau(s) = \omega_2^3(\tfrac{\partial}{\partial s}).$$

$\tau(s)$ is called the *torsion* of x at s. Then the structure equations for de_2, de_3 become

$$e_2'(s) = \sum_{j=1}^{3} e_j(s)\,\omega_2^j(\tfrac{\partial}{\partial s}) = -\kappa(s)\,e_1 + \tau(s)\,e_3$$

$$e_3'(s) = \sum_{j=1}^{3} e_j(s)\,\omega_3^j(\tfrac{\partial}{\partial s}) = -\tau(s)\,e_2.$$

Thus we have the familiar *Frenet equations*:

$$\begin{bmatrix} e_1'(s) & e_2'(s) & e_3'(s) & x'(s) \end{bmatrix} = \begin{bmatrix} e_1(s) & e_2(s) & e_3(s) & x(s) \end{bmatrix} \begin{bmatrix} 0 & -\kappa(s) & 0 & 1 \\ \kappa(s) & 0 & -\tau(s) & 0 \\ 0 & \tau(s) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the matrix on the right multiplied by the 1-form ds is equal to $\tilde{x}(s)^{-1}d(\tilde{x}(s))$, and so is exactly the pullback of the Maurer-Cartan form on $E(3)$ via $\tilde{x}$.

Applying Important Lemma 1 yields the following theorem:

Theorem: Two nondegenerate curves $x_1, x_2 : I \to \mathbb{E}^3$ parametrized by arc length differ by a rigid motion if and only if they have the same curvature $\kappa(s)$ and torsion $\tau(s)$.

3. CURVES IN $\mathbb{A}^3$

Now consider a smooth parametrized curve $x : I \to \mathbb{A}^3$ which maps some open interval $I \subset \mathbb{R}$ into affine space. A frame along x is a choice, for each $t \in I$, of a unimodular basis $\{e_1(t), e_2(t), e_3(t)\}$ for the tangent space $T_{x(t)}\mathbb{A}^3$. The situation is quite different from that of Euclidean space; for

instance, the Euclidean notion of arc length of a curve is not invariant under the action of $A(3)$. Moreover, we have much greater freedom in choosing our frame; the only requirement is that $\det[e_1 \ e_2 \ e_3] = 1$.

In the Euclidean case we used the first derivative of x to choose e_1 and the second derivative to choose e_2, stopping along the way to normalize so that the frame would be orthonormal. This determined e_3 essentially uniquely, but from the structure equations it is clear that e_3 depends on the third derivative of x. In order for this procedure to work, we had to assume that x was nondegenerate, i.e., that the vectors $x'(s), x''(s)$ were linearly independent for each $s \in I$.

Since orthonormality is not required in the affine case, our first attempt at an adapted frame might be

$$e_1(t) = x'(t)$$
$$e_2(t) = x''(t)$$
$$e_3(t) = x'''(t).$$

In order for this to work, we must assume that the vectors $x'(t), x''(t), x'''(t)$ are linearly independent for each $t \in I$; such a curve will be called *nondegenerate*. For nondegenerate curves the only problem with this choice of frame is that it is not necessarily unimodular. But we can fix this by defining the adapted frame to be

$$e_1(t) = \frac{x'(t)}{\sqrt[3]{\det[x'(t) \ x''(t) \ x'''(t)]}}$$
$$e_2(t) = \frac{x''(t)}{\sqrt[3]{\det[x'(t) \ x''(t) \ x'''(t)]}}$$
$$e_3(t) = \frac{x'''(t)}{\sqrt[3]{\det[x'(t) \ x''(t) \ x'''(t)]}}.$$

(Exercise: why is this frame invariant under the action of $A(3)$?) Now wouldn't it be nice to get rid of that ugly denominator? Suppose we reparametrize the curve by setting $x(s) = x(t(s))$ for some invertible function $t(s)$. Then

$$\frac{dx}{ds} = t'(s)\frac{dx}{dt}$$

$$\frac{d^2x}{ds^2} \equiv t'(s)^2 \frac{d^2x}{dt^2} \quad \mod \frac{dx}{ds}$$

$$\frac{d^3x}{ds^3} \equiv t'(s)^3 \frac{d^3x}{dt^3} \quad \mod \{\frac{dx}{ds}, \frac{d^2x}{ds^2}\}$$

so $\det[x'(s)\ x''(s)\ x'''(s)] = t'(s)^6 \det[x'(t)\ x''(t)\ x'''(t)]$. This suggests that we define

$$s(t) = \int_{t_0}^{t} \sqrt[6]{\det[x'(\tau)\ x''(\tau)\ x'''(\tau)]}\, d\tau.$$

This quantity is the *affine arc length*, and it is invariant under the action of $A(3)$. Note that unlike Euclidean arc length, which depends only on the first derivative of x, the affine arc length depends on the third derivative of x. In fact, this is dependent on the dimension of the ambient affine space; the affine arc length of curves in $\mathbb{A}^n$ depends on the nth derivative of x.

Assuming that the curve is parametrized by affine arc length s, we have

$$e_1(s) = x'(s)$$
$$e_2(s) = x''(s)$$
$$e_3(s) = x'''(s).$$

Therefore

$$x'(s) = e_1(s)$$
$$e_1'(s) = e_2(s)$$
$$e_2'(s) = e_3(s)$$
$$e_3'(s) = \kappa_1(s)\, e_1(s) + \kappa_2(s)\, e_2(s)$$

for some functions $\kappa_1(s), \kappa_2(s)$, called the *affine curvatures* of x at s. (Exercise: Why is there no e_3 term in $e_3'(s)$?) Thus the affine analog of the Frenet equations is

$$\begin{bmatrix} e_1'(s)\ e_2'(s)\ e_3'(s)\ x'(s) \end{bmatrix} = \begin{bmatrix} e_1(s)\ e_2(s)\ e_3(s)\ x(s) \end{bmatrix} \begin{bmatrix} 0 & 0 & \kappa_1(s) & 1 \\ 1 & 0 & \kappa_2(s) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Applying Important Lemma 1 yields the following theorem:

Theorem: Two nondegenerate affine curves $x_1, x_2 : I \to \mathbb{A}^3$ parametrized by affine arc length differ by an affine transformation if and only if they have the same affine curvatures $\kappa_1(s), \kappa_2(s)$.

Exercises

1. Prove Important Lemma 1. (Hint: Define $h : \Sigma \to G$ by

$$h(x) = \tilde{f}_2(x)\tilde{f}_1(x)^{-1}$$

and show that $dh = 0$. Therefore, $h(x)$ must be constant .)

2. Prove that the choice of affine frame is invariant under the action of $A(3)$.

3. Use the structure equations of $A(3)$ to compute the pullbacks of the Maurer-Cartan forms ω^i, ω^i_j on $A(3)$ via the map $\tilde{x} : I \to A(3)$ for a nondegenerate curve x. Why is there no e_3 term in the expression for $e'_3(s)$?

4. Consider the case of a curve $x : I \to \mathbb{A}^2$. How would you define an adapted frame $\tilde{x} : I \to A(2)$ along x? How would you define nondegeneracy for x? How would you define affine arc length for x? Use the structure equations of the Maurer-Cartan forms on $A(2)$ to find a complete set of invariants for affine curves $x : I \to \mathbb{A}^2$ parametrized by affine arc length.

5. Hopefully you discovered a single invariant $\kappa(s)$, called the *affine curvature* of x, in Exercise 4. Suppose that $\kappa = \kappa(s)$ is constant. Show that

a) If $\kappa = 0$, then x is a straight line or a parabola.

b) If $\kappa > 0$, then x is a hyperbola.

c) If $\kappa < 0$, then x is a circle or an ellipse.

(Hint: In each case, you should be able to show that $x(s)$ satisfies a differential equation whose general solution is not difficult to find. Since each of these conditions is preserved under affine transformations, you can perform an affine transformation to eliminate most of the arbitrary constants in the general solution. Finally, eliminate the parameter s to show that x lies on the appropriate conic section.)