# LECTURE 4: SURFACES IN EUCLIDEAN AND AFFINE SPACES

## 1. INTRODUCTION

In the previous section, we saw that a curve $x : I \to \mathbb{E}^3$ parametrized by arc length $s$ is completely determined up to rigid motions of $\mathbb{E}^3$ by its curvature $\kappa(s)$ and torsion $\tau(s)$. We may express this by saying that the curvature and torsion form a complete set of invariants for curves in $\mathbb{E}^3$. In general, Important Lemma 1 tells us when we have found a complete set of invariants for a "nice" submanifold $f : \Sigma \to G/H$: assuming that there is a canonical, invariant way of choosing a lifting $\tilde{f} : \Sigma \to G$ (this is what "nice" means), a complete set of invariants is contained in $\tilde{f}^*\omega$, the pullback via $\tilde{f}$ of the Maurer-Cartan form $\omega$ of $G$.

One question we didn't address was whether the functions $\kappa$ and $\tau$ could be prescribed arbitrarily. In other words, given arbitrary functions $\kappa(s), \tau(s)$, does there necessarily exist a curve $x : I \to \mathbb{E}^3$ which is parametrized by arc length and has curvature $\kappa(s)$ and torsion $\tau(s)$? The answer is yes, but this result is particular to one-dimensional submanifolds of homogenous spaces $G/H$. It follows from the following lemma:

**Important Lemma 2:** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, and suppose that $\varphi$ is a $\mathfrak{g}$-valued 1-form on a connected and simply connected manifold $\Sigma$. Then there exists a smooth map $\tilde{f} : \Sigma \to G$ with $\tilde{f}^*\omega = \varphi$ if and only if $\varphi$ satisfies the equation

$$d\varphi = -\varphi \wedge \varphi.$$

When $\Sigma$ is a curve, this condition is automatically satisfied since any 2-form vanishes on $\Sigma$. But when $\Sigma$ is a surface it will give compatibility conditions that must be satisfied in order for a surface with given invariants to exist.

## 2. SURFACES IN $\mathbb{E}^3$

Consider a smooth, embedded surface $x : \Sigma \to \mathbb{E}^3$, where $\Sigma$ is an open set in $\mathbb{R}^2$. (For ease of notation, we will not distinguish between $\Sigma$ and its image $x(\Sigma)$.) We want to choose, for each point $x \in \Sigma$, an orthonormal frame $\{e_1(x), e_2(x), e_3(x)\}$ for the tangent space $T_x\mathbb{E}^3$.

Since $x$ is an embedding, there is a well-defined tangent plane $T_x\Sigma$ for each point $x \in \Sigma$, so we can make our first adaptation by requiring that $e_3(x)$ be

orthogonal to $T_x\Sigma$. This choice is clearly invariant under the action of $E(3)$. $e_1(x)$ and $e_2(x)$ will then form a basis for $T_x\Sigma$; for now we allow $e_1(x), e_2(x)$ to be an arbitrary orthonormal basis of $T_x\Sigma$.

Now consider the Maurer-Cartan forms of $E(3)$ restricted to this frame. Since the tangent plane of $T_x\Sigma$ is spanned by $e_1(x), e_2(x)$, the vector-valued 1-form $dx$ must be a linear combination of $e_1$ and $e_2$. From the structure equation

$$dx = \sum_{i=1}^{3} e_i \, \omega^i$$

we see that we must have $\omega^3 = 0$ on this frame, and the linearly independent 1-forms $\omega^1, \omega^2$ form a basis for the space of 1-forms on $\Sigma$. In fact, the first fundamental form of $\Sigma$ is given by

$$I = \langle dx, dx \rangle = (\omega^1)^2 + (\omega^2)^2.$$

Now since $\omega^3 = 0$, we must also have $d\omega^3 = 0$. Therefore

$$0 = d\omega^3 = -\omega_1^3 \wedge \omega^1 - \omega_2^3 \wedge \omega^2.$$

Since $\omega^1, \omega^2$ are linearly independent 1-forms, *Cartan's Lemma* states that there exist functions $h_{11}, h_{12}, h_{22}$ such that

$$\begin{bmatrix} \omega_1^3 \\ \omega_2^3 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}.$$

The second fundamental form of $\Sigma$ is given by

$$II = \omega_1^3 \, \omega^1 + \omega_2^3 \, \omega^2 = h_{11}(\omega^1)^2 + 2h_{12}\omega^1\omega^2 + h_{22}(\omega^2)^2.$$

In order to make our next frame adaptation, we will compute how the matrix $[h_{ij}]$ varies if we choose a different frame. So suppose that $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ is any other orthonormal frame of the form

$$\begin{bmatrix} \tilde{e}_1 & \tilde{e}_2 & \tilde{e}_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Set $B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Computing the Maurer-Cartan form of the new frame shows that

$$\begin{bmatrix} \tilde{\omega}^1 \\ \tilde{\omega}^2 \end{bmatrix} = B^{-1} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}, \qquad \begin{bmatrix} \tilde{\omega}_1^3 \\ \tilde{\omega}_2^3 \end{bmatrix} = {}^t\!B \begin{bmatrix} \omega_1^3 \\ \omega_2^3 \end{bmatrix}$$

and therefore,

$$\begin{bmatrix} \tilde{h}_{11} & \tilde{h}_{12} \\ \tilde{h}_{12} & \tilde{h}_{22} \end{bmatrix} = {}^t\!B \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix} B.$$

Since any symmetric matrix can be transformed to a diagonal matrix by such a change of basis, we can choose $e_1(x), e_2(x)$ so that

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix} = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}.$$

We will assume that $\kappa_1 \neq \kappa_2$ at every point of $\Sigma$. This is equivalent to assuming that $\Sigma$ has no umbilic points, and in this case the condition that $[h_{ij}]$ be diagonal determines $e_1(x), e_2(x)$ essentially uniquely. This condition is also invariant under the action of $E(3)$. In this basis the second fundamental form of $\Sigma$ is

$$II = \kappa_1 (\omega^1)^2 + \kappa_2 (\omega^2)^2.$$

$\kappa_1$ and $\kappa_2$ are called the *principal curvatures* of $\Sigma$ at $x$.

We will now show that any surface without umbilic points is determined up to rigid motions by its first and second fundamental forms. (This is actually true even if $\Sigma$ has umbilic points, but the proof is slightly more involved.) Suppose that $x, \bar{x} : \Sigma \to \mathbb{E}^3$ have the same first and second fundamental forms. Then it is clear that

$$\omega^1 = \bar{\omega}^1, \qquad \omega^2 = \bar{\omega}^2,$$

and $\omega^3 = \bar{\omega}^3 = 0$. Therefore

$$d\omega^1 = d\bar{\omega}^1$$

which implies that

$$(\omega_2^1 - \bar{\omega}_2^1) \wedge \omega^2 = 0.$$

By Cartan's lemma, $\omega_2^1 - \bar{\omega}_2^1$ must be a multiple of $\omega^2$. But by the same reasoning, the fact that $d\omega^2 = d\bar{\omega}^2$ implies that $\omega_2^1 - \bar{\omega}_2^1$ must also be a multiple of $\omega^1$. Since $\omega^1$ and $\omega^2$ are linearly independent, we must have $\omega_2^1 = \bar{\omega}_2^1$. The fact that $II = \bar{II}$ implies that $\omega_1^3 = \bar{\omega}_1^3$ and $\omega_2^3 = \bar{\omega}_2^3$. Now Important Lemma 1 yields the desired result:

**Theorem:** Two embedded surfaces $x_1, x_2 : \Sigma \to \mathbb{E}^3$ without umbilic points differ by a rigid motion if and only if they have the same first and second fundamental forms.

Now we consider the question discussed in the introduction, namely, can the first and second fundamental forms be prescribed arbitrarily? In other words, given 1-forms $\omega^1, \omega^2, \omega_1^3, \omega_2^3$ on a surface $\Sigma$, what conditions must these forms satisfy in order that there exist an embedding $x : \Sigma \to \mathbb{E}^3$ whose first and second fundamental forms are

$$I = (\omega^1)^2 + (\omega^2)^2$$
$$II = \omega_1^3 \omega^1 + \omega_2^3 \omega^2?$$

Important Lemma 2 gives the answer: the forms must satisfy the structure equations of the Maurer-Cartan forms on $E(3)$. The first three of these equations are

$$d\omega^1 = -\omega^1_2 \wedge \omega^2$$
$$d\omega^2 = \omega^1_2 \wedge \omega^1$$
$$d\omega^3 = 0 = -\omega^3_1 \wedge \omega^1 - \omega^3_2 \wedge \omega^2.$$

The first two equations uniquely determine the form $\omega^1_2$, called the *Levi-Civita connection form* of the metric given by the first fundamental form $I = (\omega^1)^2 + (\omega^2)^2$. The third equation says that $\omega^3_1$ and $\omega^3_2$ must be symmetric linear combinations of $\omega^1$ and $\omega^2$.

The remaining structure equations are

$$d\omega^1_2 = \omega^3_1 \wedge \omega^3_2$$
$$d\omega^3_1 = \omega^3_2 \wedge \omega^1_2$$
$$d\omega^3_2 = -\omega^3_1 \wedge \omega^1_2.$$

The first of these equations is called the *Gauss equation*, and the last two are called the *Codazzi equations*. By Important Lemma 2, we have the following theorem:

**Theorem:** Suppose that the forms $\omega^1, \omega^2, \omega^3_1, \omega^3_2$ on $\Sigma$ together with the Levi-Civita connection form $\omega^1_2$ determined by $\omega^1$ and $\omega^2$ satisfy the Gauss and Codazzi equations. Then there exists an immersed surface $x : \Sigma \to \mathbb{E}^3$, unique up to rigid motion, whose first and second fundamental forms are

$$I = (\omega^1)^2 + (\omega^2)^2$$
$$II = \omega^3_1 \omega^1 + \omega^3_2 \omega^2.$$

## 3. Surfaces in $\mathbb{A}^3$

Now consider a smooth, embedded surface $x : \Sigma \to \mathbb{A}^3$. In order to compute invariants for such a surface, we need to use the geometry of the surface to construct a unimodular frame $\{e_1(x), e_2(x), e_3(x)\}$ for the tangent space $T_x\mathbb{A}^3$ for each $x \in \Sigma$.

In the Euclidean case we began by choosing $e_3$ to be orthogonal to the tangent plane $T_x\Sigma$; it followed that $e_1, e_2$ must be a basis for $T_x\Sigma$. In affine space there is no notion of orthogonality, so we cannot normalize $e_3$ immediately. However, we can still make our first adaptation by requiring that $e_1, e_2$ span the tangent plane $T_x\Sigma$. This condition is clearly invariant under the action of $A(3)$. When we restrict the Maurer-Cartan form to such a frame, the same reasoning as in the Euclidean case tells us that the linearly independent 1-forms $\omega^1, \omega^2$ form a basis for the space of 1-forms on $\Sigma$ and

that $\omega^3 = d\omega^3 = 0$. Therefore there exist functions $h_{11}, h_{12}, h_{22}$ such that

$$\begin{bmatrix} \omega_1^3 \\ \omega_2^3 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}.$$

In order to make our next frame adaptation, we will compute how the matrix $[h_{ij}]$ varies if we choose a different frame. So suppose that $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ is any other affine frame with the property that $\tilde{e}_1, \tilde{e}_2$ span the tangent space $T_x\Sigma$. Then $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ has the form

$$\begin{bmatrix} \tilde{e}_1 & \tilde{e}_2 & \tilde{e}_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} & & r_1 \\ & B & r_2 \\ 0 & 0 & (\det B)^{-1} \end{bmatrix}$$

where $B \in GL(2)$. Computing the Maurer-Cartan form of the new frame shows that

$$\begin{bmatrix} \tilde{\omega}^1 \\ \tilde{\omega}^2 \end{bmatrix} = B^{-1} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}, \qquad \begin{bmatrix} \tilde{\omega}_1^3 \\ \tilde{\omega}_2^3 \end{bmatrix} = (\det B)\, {}^tB \begin{bmatrix} \omega_1^3 \\ \omega_2^3 \end{bmatrix}$$

and therefore,

$$\begin{bmatrix} \tilde{h}_{11} & \tilde{h}_{12} \\ \tilde{h}_{12} & \tilde{h}_{22} \end{bmatrix} = (\det B)\, {}^tB \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix} B.$$

This transformation has the property that $\det[\tilde{h}_{ij}] = (\det B)^4 \det[h_{ij}]$, so the sign of the determinant is fixed. We will assume that $\det[h_{ij}] > 0$; in this case the surface is said to be *elliptic*. Then we can choose the matrix $B$ so that $[h_{ij}]$ is the identity matrix. This determines the frame up to a transformation of the form

$$\begin{bmatrix} \tilde{e}_1 & \tilde{e}_2 & \tilde{e}_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} & & r_1 \\ & B & r_2 \\ 0 & 0 & 1 \end{bmatrix}$$

with $B \in SO(2)$.

The quadratic form

$$I = \omega_1^3\, \omega^1 + \omega_2^3\, \omega^2 = (\omega^1)^2 + (\omega^2)^2$$

is now well-defined. It is called the *affine first fundamental form*, and it defines a metric on $\Sigma$ which is invariant under the action of $A(3)$.

We still don't have a well-defined normal vector for $\Sigma$ because we still allow changes of frame with

$$\tilde{e}_3 = e_3 + r_1 e_1 + r_2 e_2.$$

Now consider the connection form $\omega_3^3$. Under a change of frame as above, we can compute that

$$\tilde{\omega}_3^3 = \omega_3^3 + r_1\omega^1 + r_2\omega^2.$$

Since $\omega^1, \omega^2$ are linearly independent, there is a unique choice of $r_1, r_2$ for which $\omega_3^3 = 0$. In particular, imposing this condition yields a unique, affine-invariant choice of $e_3$, called the *affine normal* of $\Sigma$.

Since we now have $\omega_3^3 = 0$, we must have $d\omega_3^3 = 0$. Therefore

$$0 = d\omega_3^3$$
$$= -\omega_1^3 \wedge \omega_3^1 - \omega_2^3 \wedge \omega_3^2$$
$$= \omega_3^1 \wedge \omega^1 + \omega_3^2 \wedge \omega^2.$$

By Cartan's lemma, there exist functions $\ell_{11}, \ell_{12}, \ell_{22}$ such that

$$\begin{bmatrix} \omega_3^1 \\ \omega_3^2 \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{12} & \ell_{22} \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}.$$

It is not difficult to show that the quadratic form

$$II = \omega_3^1\, \omega^1 + \omega_3^2\, \omega^2 = \ell_{11}(\omega^1)^2 + 2\ell_{12}\, \omega^1\, \omega^2 + \ell_{22}(\omega^2)^2$$

is well-defined; this is the *affine second fundamental form*. The quantity $L = \frac{1}{2}(\ell_{11} + \ell_{22})$ is called the *affine mean curvature* of $\Sigma$. It is identically zero if and only if $\Sigma$ is a critical point of the affine area functional, just as in the Euclidean case.

There are still more invariants; differentiating the equations

$$\omega_1^3 = \omega^1, \qquad \omega_2^3 = \omega^2$$

and using Cartan's Lemma shows that there exist functions $h_{111}, h_{112}, h_{122}, h_{222}$ such that

$$\begin{bmatrix} 2\omega_1^1 \\ \omega_2^1 + \omega_1^2 \\ 2\omega_2^2 \end{bmatrix} = \begin{bmatrix} h_{111} & h_{112} \\ h_{112} & h_{122} \\ h_{122} & h_{222} \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}.$$

Furthermore, since we now have $\omega_1^1 + \omega_2^2 = 0$ (Exercise: why?), it follows that

$$h_{122} = -h_{111}, \qquad h_{112} = -h_{222}.$$

The cubic form

$$P = \sum_{i,j,k=1}^{2} h_{ijk}\omega^i \omega^j \omega^k$$

is called the *Fubini-Pick form* of $\Sigma$. Together with the first and second affine fundamental forms, it forms a complete set of invariants for elliptic affine surfaces.

## Exercises

1. Any invariant of a surface $\Sigma \subset \mathbb{E}^3$ that can be expressed purely in terms of the first fundamental form

$$I = (\omega^1)^2 + (\omega^2)^2$$

is called an *intrinsic* invariant of the surface. For instance, arc length and area are intrinsic quantities on $\Sigma$. The principal curvatures $\kappa_1, \kappa_2$, however, are not; they depend not only on the metric, but also on how the surface is embedded.

The function $K = \kappa_1 \kappa_2$ on $\Sigma$ is called the *Gauss curvature* of $\Sigma$. Even though $\kappa_1, \kappa_2$ are not intrinsic quantities, Gauss' Theorem Egregium states that their product $K$ is in fact intrinsic. We will prove this in several steps.

The 1-forms $\omega^1, \omega^2$ are determined by the first fundamental form up to a transformation of the form

$$\begin{bmatrix} \tilde{\omega}^1 \\ \tilde{\omega}^2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}.$$

a) Show that the area form

$$dA = \omega^1 \wedge \omega^2$$

is an intrinsic quantity.

b) Show that if $\omega^1_2$ is the Levi-Civita connection form corresponding to $\{\omega^1, \omega^2\}$, then

$$\tilde{\omega}^1_2 = \omega^1_2 + d\theta.$$

Conclude that $d\omega^1_2$ is an intrinsic quantity.

c) Show that $d\omega^1_2 = \omega^3_1 \wedge \omega^3_2 = K\,\omega^1 \wedge \omega^2$. Conclude that $K$ must be an intrinsic quantity.

2. In terms of local coordinates $(u, v)$ on a surface $x : \Sigma \to \mathbb{E}^3$, the first and second fundamental forms are usually written

$$I = E\,du^2 + 2F\,du\,dv + G\,dv^2$$
$$II = e\,du^2 + 2f\,du\,dv + g\,dv^2$$

It is always possible to parametrize a surface locally so that the coordinate curves are principal curves on the surface, and for any such parametrization $F = f = 0$.

a) For a parametrization with $F = f = 0$, show that the frame

$$e_1 = \frac{1}{\sqrt{E}}\,x_u, \qquad e_2 = \frac{1}{\sqrt{G}}\,x_v, \qquad e_3 = e_1 \times e_2$$

is orthonormal, and that its dual forms are

$$\omega^1 = \sqrt{E}\,du, \qquad \omega^2 = \sqrt{G}\,dv, \qquad \omega^3 = 0.$$

b) Use the structure equations for $d\omega^1$, $d\omega^2$ to show that

$$\omega^1_2 = \frac{1}{2\sqrt{EG}}\,(E_v\,du - G_u\,dv).$$

c) Use the second fundamental form to show that

$$\omega^3_1 = \frac{e}{E}\,\omega^1 = \frac{e}{\sqrt{E}}\,du$$

$$\omega^3_2 = \frac{g}{G}\,\omega^1 = \frac{g}{\sqrt{G}}\,dv$$

d) Show that the Gauss equation is equivalent to

$$\frac{eg}{EG} = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right].$$

The left-hand side is, by definition, the Gauss curvature $K$, and this equation shows that $K$ is in fact completely determined by the coefficients of the first fundamental form.

e) Show that the Codazzi equations are equivalent to

$$e_v = \frac{1}{2}E_v\left(\frac{e}{E} + \frac{g}{G}\right)$$

$$g_u = \frac{1}{2}G_u\left(\frac{e}{E} + \frac{g}{G}\right).$$

3. Let $x : \Sigma \to \mathbb{A}^3$ be an elliptic affine surface, and suppose that the affine second fundamental form is a multiple of the affine first fundamental form, so that

$$\omega^1_3 = \lambda\,\omega^1, \qquad \omega^2_3 = \lambda\,\omega^2$$

for some function $\lambda$.

a) Prove that $\lambda$ is constant. (Hint: use the structure equations to differentiate the equations above and use Cartan's lemma.)

b) Show that if $\lambda = 0$, then $de_3 = 0$, and therefore the affine normals of $\Sigma$ are all parallel. Such surfaces are called *improper affine spheres*.

c) Show that if $\lambda \neq 0$, then $d(x - \frac{1}{\lambda}e_3) = 0$. Therefore, all the affine normals intersect at the point $x_0 = x - \frac{1}{\lambda}e_3$. Such surfaces are called *proper affine spheres*.