Automorphisms of Decompositions

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Introduction

Quantum logic is based on using the orthomodular lattice of closed subspaces of a Hilbert space $\mathcal{H}$ to study the quantum mechanical system associated with $\mathcal{H}$. Closed subspaces correspond to direct product decompositions

$$\mathcal{H} \simeq A \times A^{\perp}$$

Theorem The direct product decompositions of most structures $\mathbb{A}$ (universal algebras, topological spaces, etc.) form an $\textsc{Omp}$ Fact $\mathbb{A}$.

Here we study automorphisms of Fact $\mathbb{A}$ to help understand these structures, and also for applications to quantum logic.

Note the automorphisms of Fact $\mathcal{H}$ are given by the unitaries and antiunitaries of $\mathcal{H}$ (Wigner).

Basics

Definition $(P, \leq, ', 0, 1)$ is an orthomodular poset ($\textsc{Omp}$) if

1. It is a bounded poset.
2. $'$ is an order inverting period two complementation.
3. $x \leq y' \Rightarrow x \vee y$ exists (written $x \oplus y$).
4. $x \leq y' \Rightarrow x \oplus (x \oplus y)' = y$.

Definition Aut (P) is the automorphism group of an $\textsc{Omp}$ P.

Basics

Definition A binary decomposition of $\mathbb{A}$ is an isomorphism

$$f : \mathbb{A} \to \mathbb{A}_1 \times \mathbb{A}_2$$

This decomposition is equivalent to $g : \mathbb{A} \to \mathbb{B}_1 \times \mathbb{B}_2$ if there are isomorphisms from $\mathbb{A}_i$ to $\mathbb{B}_i$ making a commuting diagram.

Notation $[\,\mathbb{A} \simeq_f \mathbb{A}_1 \times \mathbb{A}_2\,]$ for the equivalence class of f.

Definition Fact $\mathbb{A}$ is the set of equivalence classes of binary decompositions of a structure $\mathbb{A}$.

Basics

Definition $[\mathbb{A} \simeq \mathbb{A}_1 \times \mathbb{A}_2]^\perp = [\mathbb{A} \simeq \mathbb{A}_2 \times \mathbb{A}_1]$

Definition $[\mathbb{A} \simeq \mathbb{A}_1 \times \mathbb{A}_2] \leq [\mathbb{A} \simeq \mathbb{B}_1 \times \mathbb{B}_2]$ iff

1. $[\mathbb{A} \simeq \mathbb{A}_1 \times \mathbb{A}_2] = [\mathbb{A} \simeq \mathbb{C}_1 \times (\mathbb{C}_2 \times \mathbb{C}_3)]$
2. $[\mathbb{A} \simeq \mathbb{B}_1 \times \mathbb{B}_2] = [\mathbb{A} \simeq (\mathbb{C}_1 \times \mathbb{C}_2) \times \mathbb{C}_3)]$

For some ternary decomposition $\mathbb{A} \simeq \mathbb{C}_1 \times \mathbb{C}_2 \times \mathbb{C}_3$.

Theorem If $\mathbb{A}$ is a set, group, vector space, universal algebra, topological space, uniform space, etc., then Fact $\mathbb{A}$ is an OMP.

Basics

Notes:

- Many standard ways to make $\textsc{Omp}$ are special cases of this.
- Such $\textsc{Omp}$s Fact $\mathbb{A}$ are regular.
- Not all $\textsc{Omp}$s are embeddable into some Fact $\mathbb{A}$, but known examples of $\textsc{Omp}$s not embeddable into some Fact $\mathbb{A}$ coincide with those known not to be embeddable into some $\textsc{Oml}$.

Aim Further understand Fact $\mathbb{A}$ by studying its automorphism group. Start with $\mathbb{A}$ a f.d. vector space or finite set.

The f.d. vector space setting

Here Fact $\mathbb{V}$ has an easier description.

Definition For a bounded modular lattice L let $L^{(2)}$ be all ordered pairs of complementary elements of L. Define

1. $(a_1, a_2)' = (a_2, a_1)$.
2. $(a_1, a_2) \leq (b_1, b_2)$ iff $a_1 \leq b_1$ and $b_2 \leq a_2$

Proposition $L^{(2)}$ is an $\textsc{Omp}$.

Proposition For a vector space $\mathbb{V}$, Fact $\mathbb{V} \simeq (\text{Sub } \mathbb{V})^{(2)}$.

The f.d. vector space setting

Assume $\mathbb{V}$ is a 3-dimensional, the arguments work in general. One-dimensional subspaces are points a and two-dimensional subspaces are lines A of a projective plane $\mathbb{P}$.

Proposition Fact $\mathbb{V}$ has height 3, and

1. atoms are pairs aA with a a point, A a line, and $a \not{\mathbb{I}}\, A$.
2. coatoms are pairs Aa with a a point, A a line, and $a \not{\mathbb{I}}\, A$.
3. $aA \leq Bb$ iff $a \,\mathbb{I}\, B$ and $b \,\mathbb{I}\, A$.

Note: These things are quite big. If $\mathbb{V} = \mathbb{Z}_2^3$, then Fact $\mathbb{V}$ has 28 atoms, 28 blocks (maximal Boolean subalgebras), each block has 3 atoms, and each atom is in 3 blocks.

The f.d. vector space setting

Key observation Atoms aA and bB of Fact $\mathbb{V}$ have at least two coatom upper bounds iff $a = b$ or $A = B$. Call such mates.

Definition For a point a and line A let

1. $X_a =$ all atoms having a for a first spot.
2. $X_A =$ all atoms having A for a second spot
3. $\mathfrak{X} = \{X_a, X_A : a \text{ is a point and } A \text{ is a line}\}$.

Lemma The X_a and X_A are the maximal sets of pairwise mates.

The f.d. vector space setting

Lemma Let a, b be points and A, B be lines.

1. X_a and X_b are disjoint.
2. X_A and X_B are disjoint.
3. X_a and X_A are disjoint iff $a \leq A$.

Definition For a subspace S let $\mathfrak{X}_S = \{X_a : a \leq S\} \cup \{X_A : S \leq A\}$.

Lemma The $\mathfrak{X}_S$ are the maximal pairwise disjoint subsets of $\mathfrak{X}$.

The f.d. vector space setting

An automorphism α of Fact $\mathbb{V}$ induces a permutation of $\mathfrak{X}$, hence a permutation σ of Sub $\mathbb{V}$ where $\sigma(S) = T$ iff $\alpha(\mathfrak{X}_S) = \mathfrak{X}_T$.

Theorem If α is an automorphism of Fact $\mathbb{V}$, either

1. σ is an automorphism of Sub $\mathbb{V}$ and $\alpha(aA) = (\sigma a)(\sigma A)$.
2. σ is an anti-automorphism of Sub $\mathbb{V}$ and $\alpha(aA) = (\sigma A)(\sigma a)$.

Corollary The automorphism group of Fact $\mathbb{V}$ is isomorphic to the group of automorphisms and anti-automorphisms of Sub $\mathbb{V}$.

The f.d. vector space setting

Remark The Fundamental Theorem of Projective Geometry allows us to characterize the automorphisms of Fact $\mathbb{V}$ in terms of semi-linear transformations on $\mathbb{V}$ and an involution of Sub $\mathbb{V}$.

Remark The main result holds also for non-Desarguesian planes considered as modular lattices.

Remark This result shows the automorphism group of Fact $\mathbb{V}$ is transitive on atoms in a very strong way. In fact, any four blocks in "general position" can be moved to any other.

The f.d. vector space setting

Our proof shows more, each order-isomorphism of Fact $\mathbb{V}$ is of the indicated form, so is compatible with the orthocomplementation. This leads the following result.

Theorem Each $\textsc{Omp}$ Fact $\mathbb{V}$ is uniquely orthocomplemented, meaning there is only one orthocomplementation compatible with its order structure.

This final result is a bit unusual. There are many (non-isomorphic) orthocomplementations on the $\textsc{Oml}$ of subspaces of $\mathbb{R}^3$.

The finite set setting

Still in progress.

Conjecture If X is a finite set whose cardinality has enough prime factors of sufficient size, then the automorphism group of Fact X is the group of permutations of X.

When $|X| = 8$ the result is not true.

When $|X| = 27$ we think it holds. Here each automorphism arises from an automorphism of the poset of regular equivalence relations

No computers. If $|X| = 27$, then Fact X has $\dfrac{27!}{9!3!} \simeq 10^{22}$ atoms.
We seek permutations of those atoms!

Measures

Definition For P an $\textsc{Omp}$ and G an abelian group, a G-valued measure on P is a map $\sigma : P \to G$ that is finitely additive, meaning

$$\sigma(x \oplus y) = \sigma(x) + \sigma(y)$$

We have several results about measures on Fact $\mathbb{V}$ when V is over a finite field, mostly relating the existence of such measure to the relationship between G and the characteristic of the field.

Thank you for listening.

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