Latest on Varieties of ℓ-Groups, Unital ℓ-Groups, and Related Things

W. Charles Holland
University of Colorado

Boulder 2013

History

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Formal Logic

History

1847 Boolean Algebra

Formal Logic

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~1900 Quantum things

Formal Logic

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1917 Łukasiewicz Multi-Valued Logic

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and MV-Algebra

1958 Chang Completeness Theorem

Formal Logic History Lattice-Ordered Groups

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Pseudo Multi Valued

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1. Darnel & Holland, More covers of the boolean variety of unital ℓ-groups. (accepted by Algebra Universalis)
2. Darnel & Holland, Minimal non-metabelian varieties of ℓ-groups which contain no nonabelian o-groups.

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Much Richer Than Varieties of ℓ -groups!!

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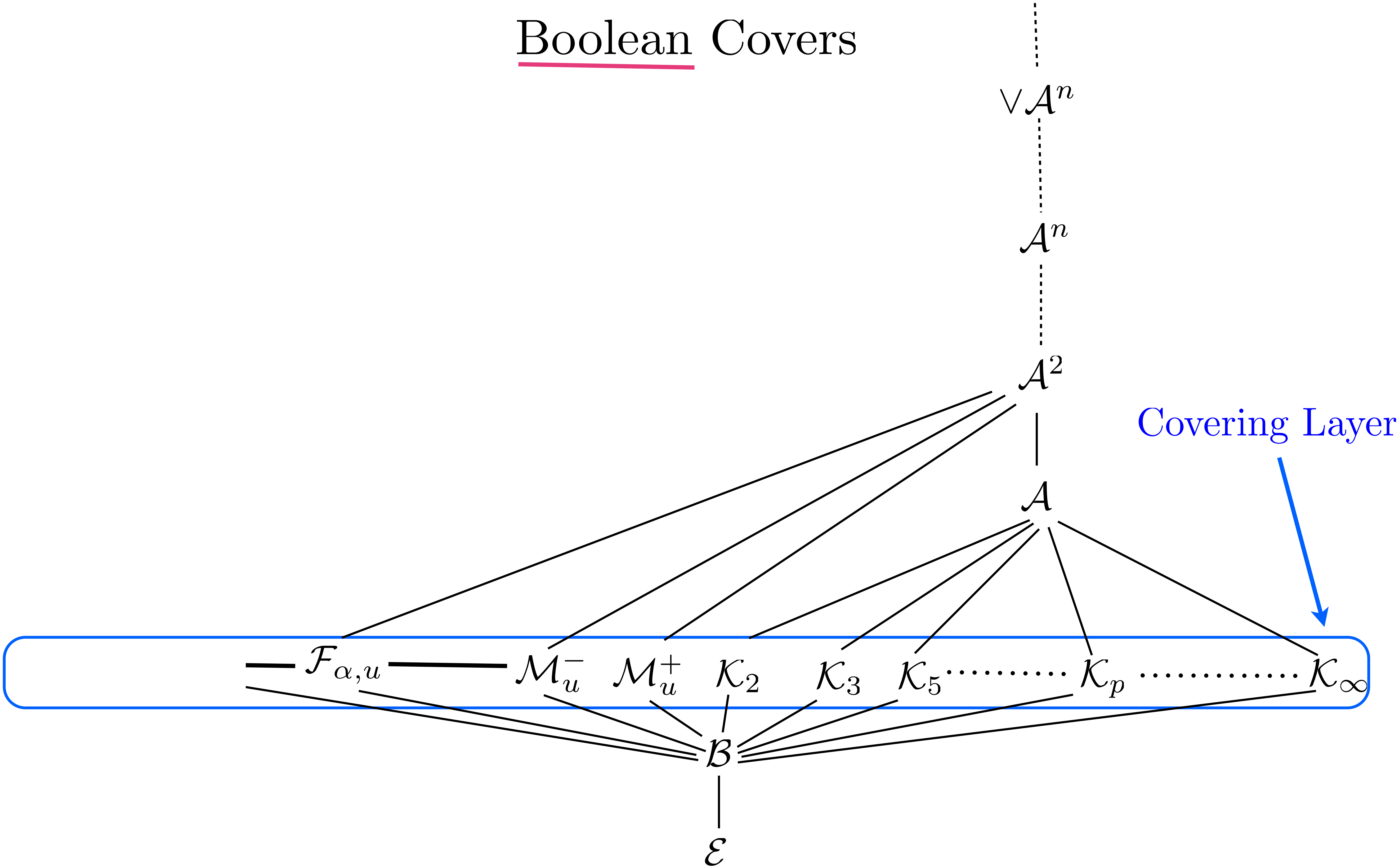

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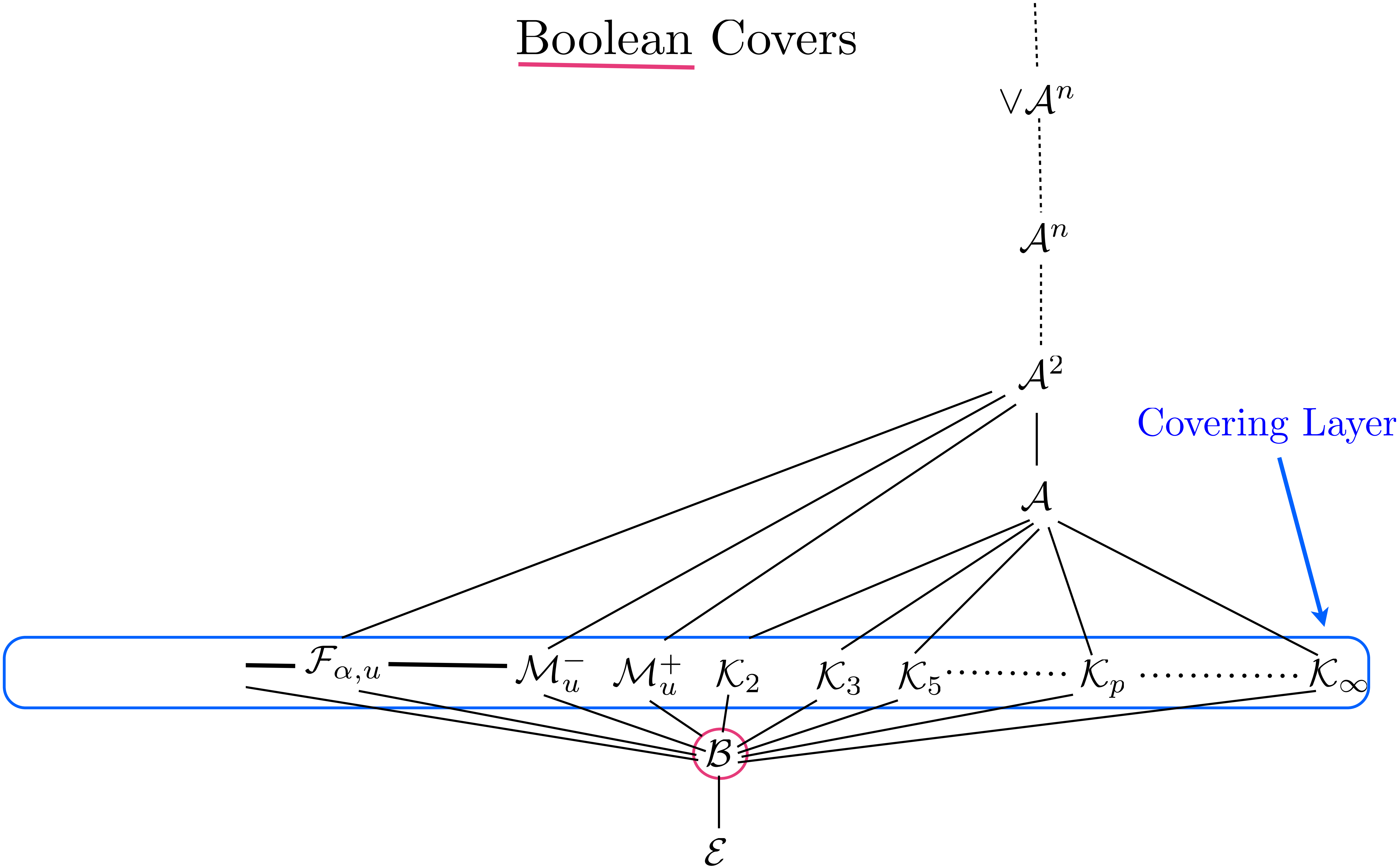

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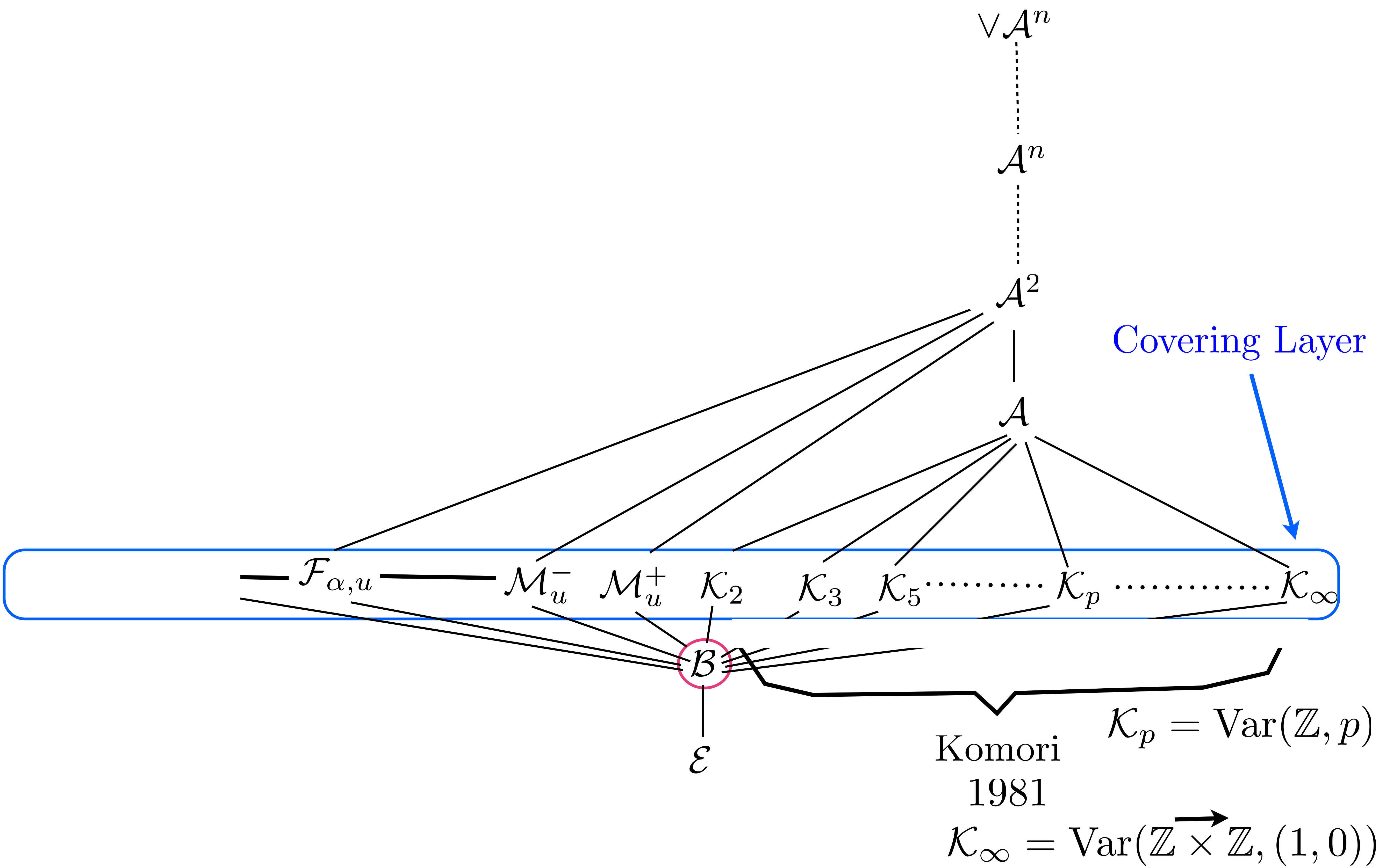

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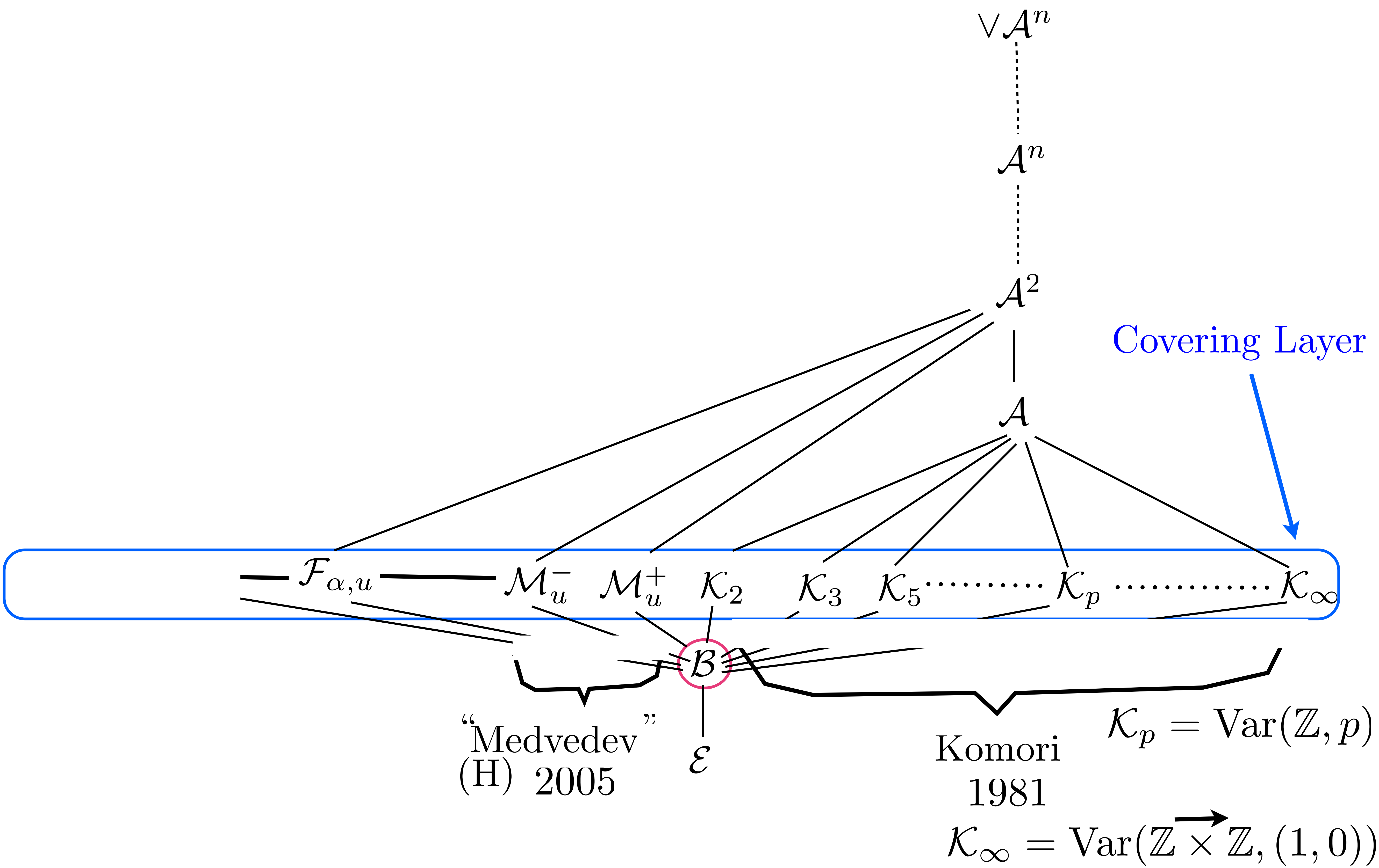

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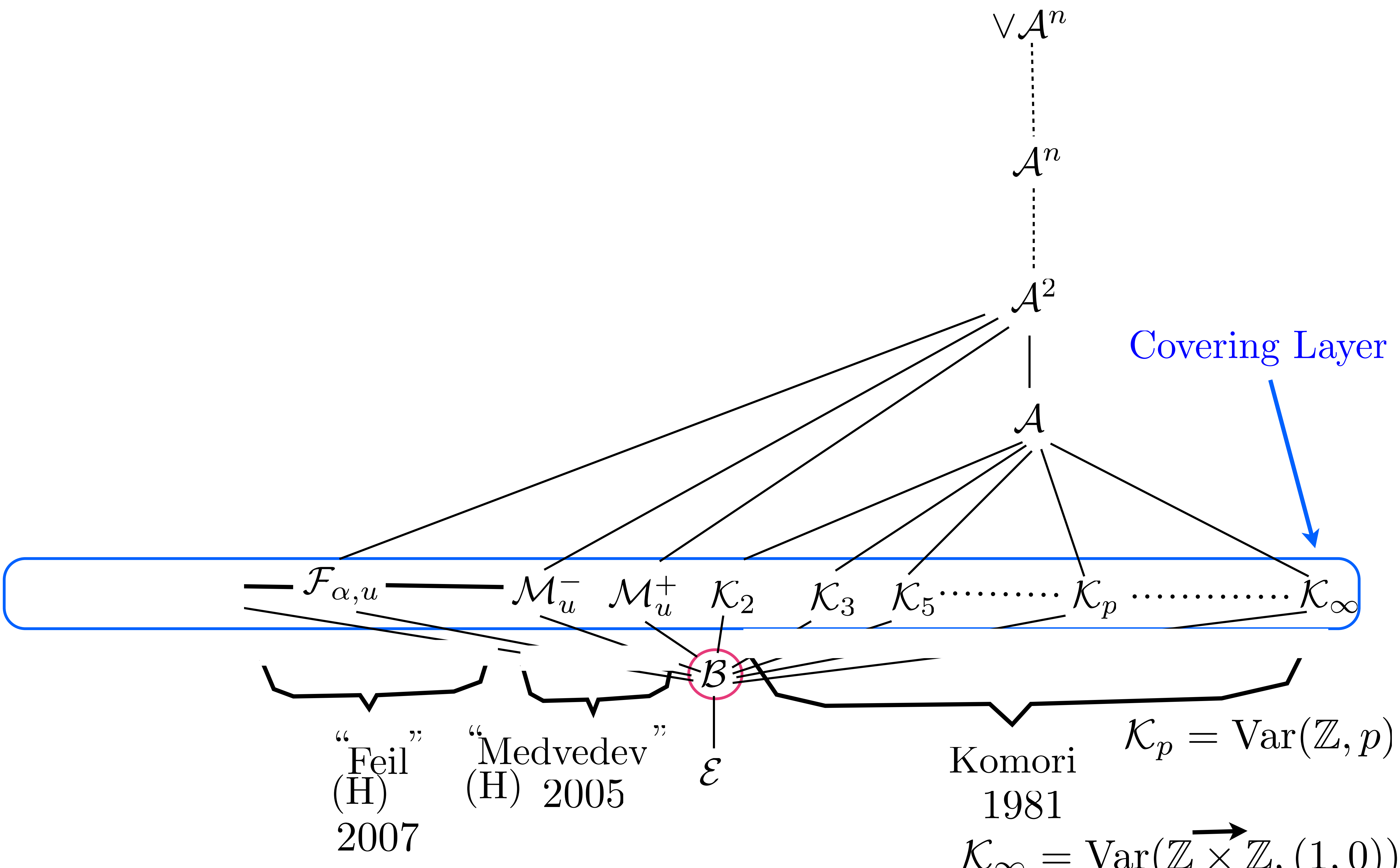

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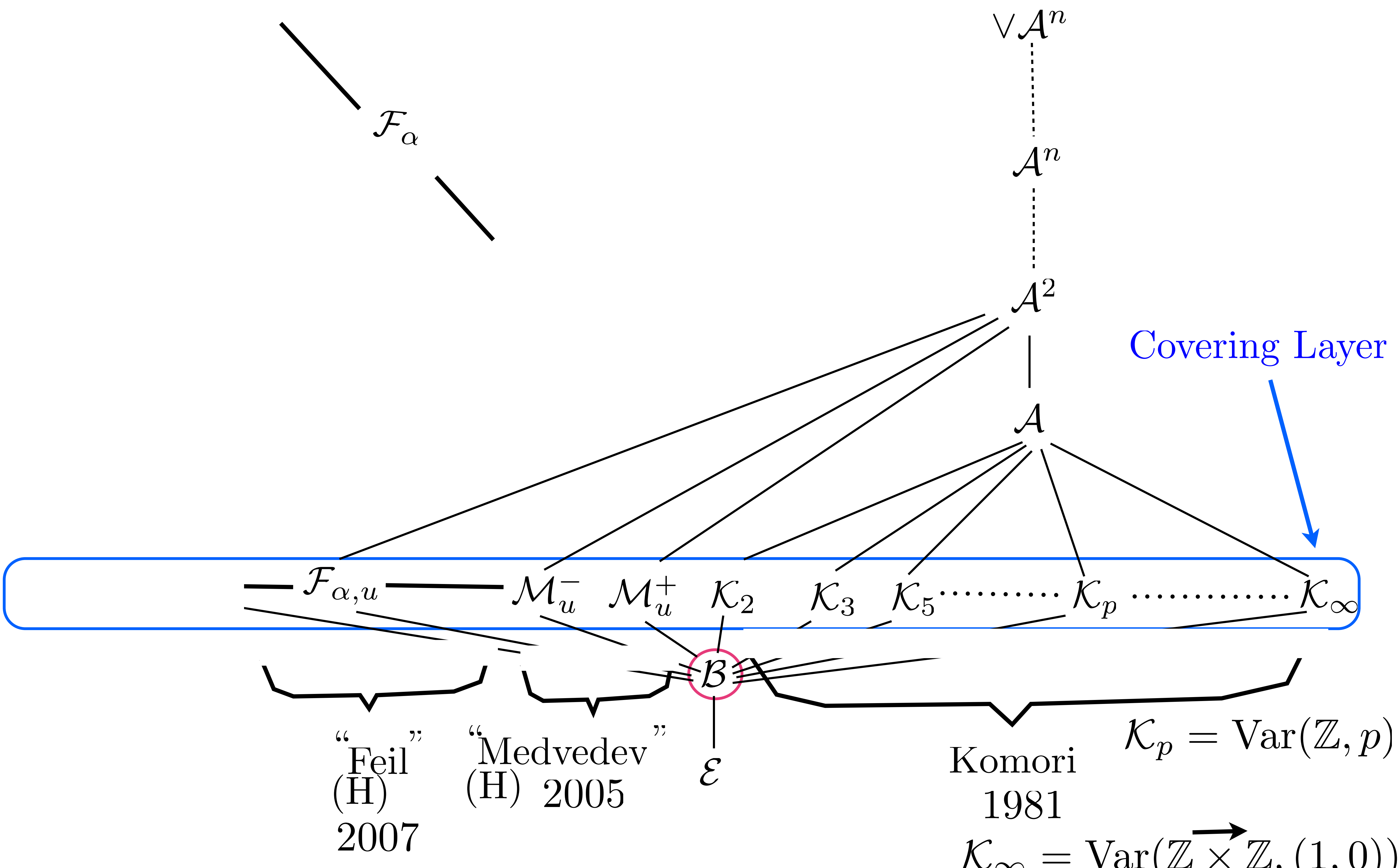

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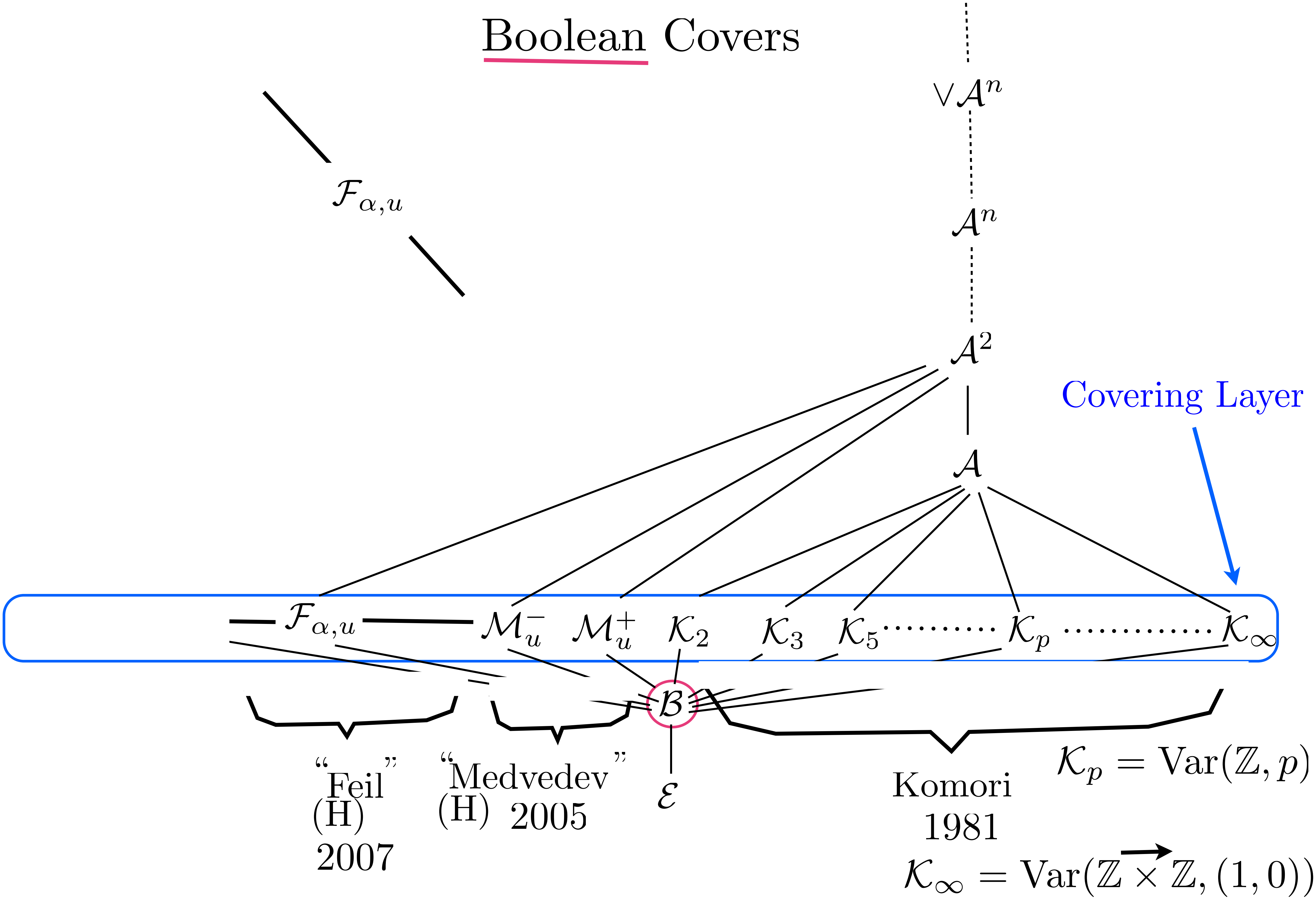

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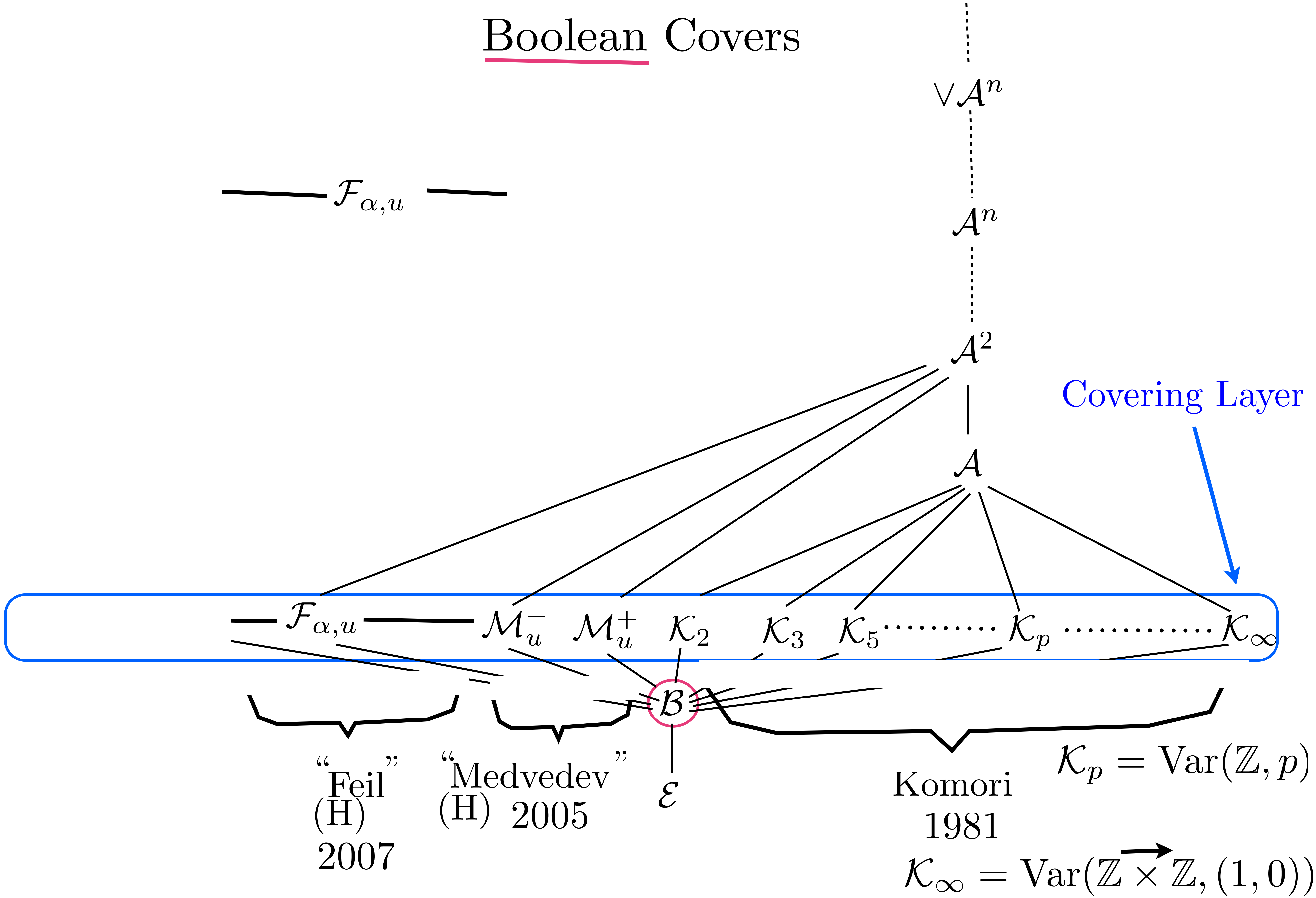

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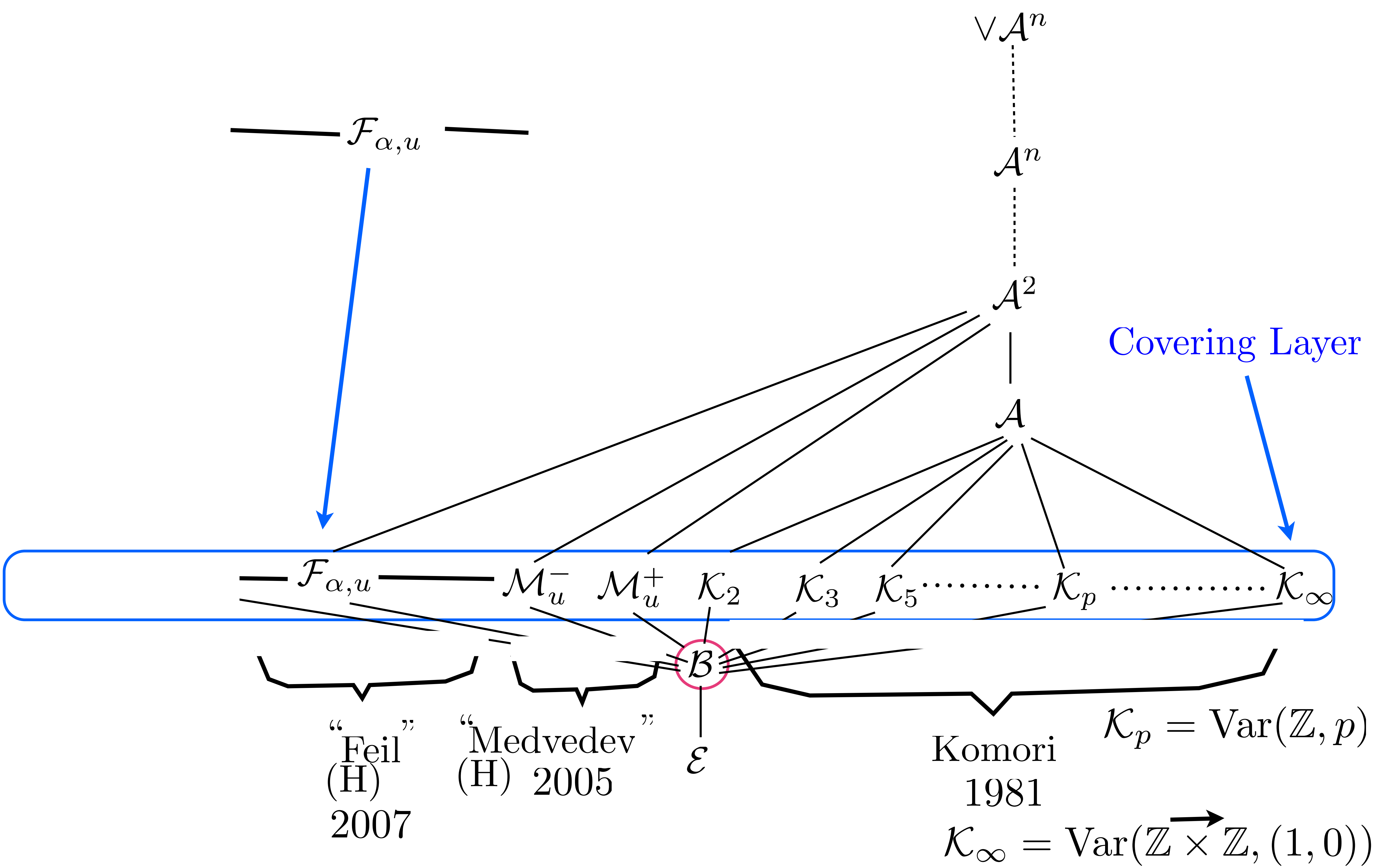

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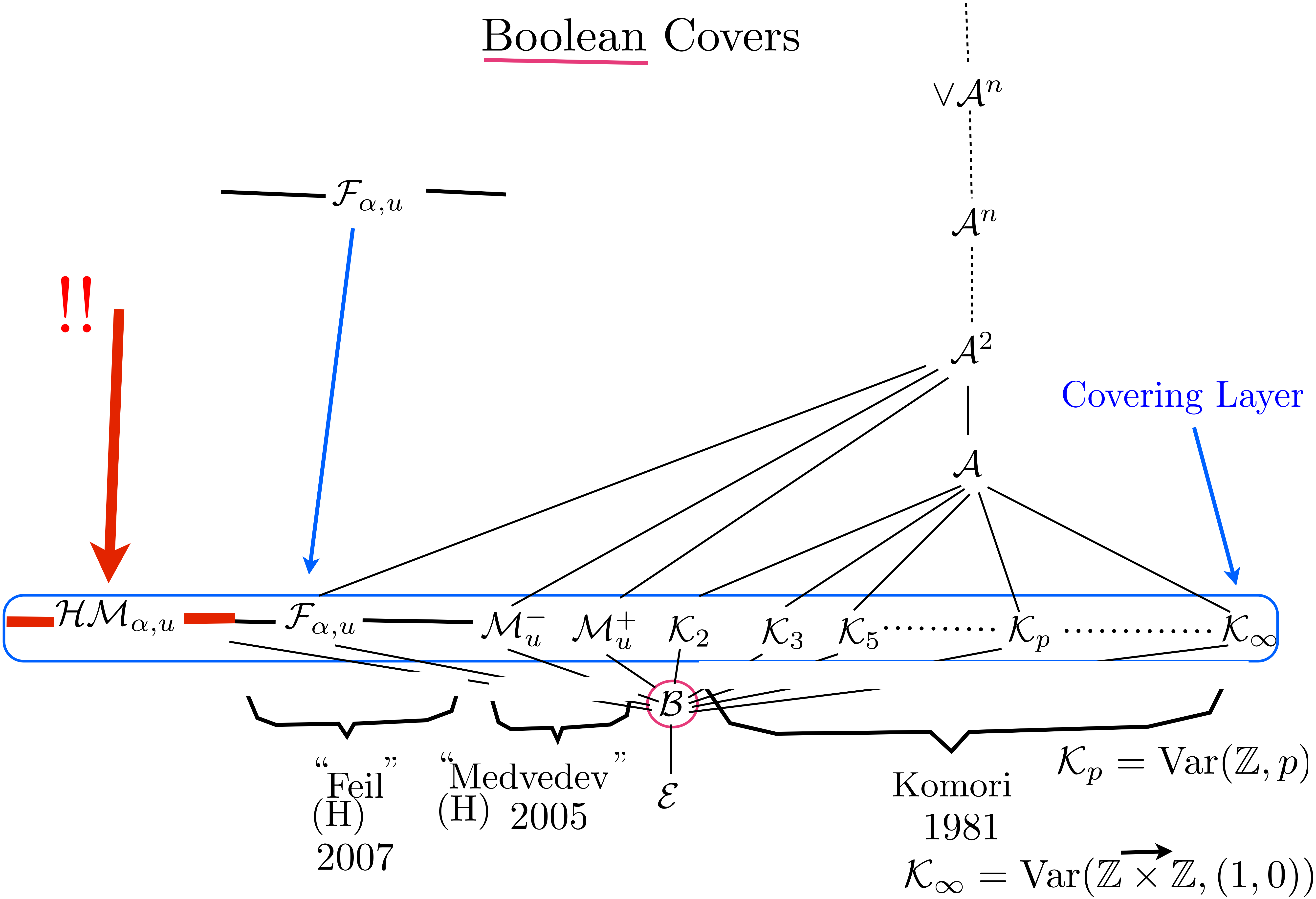

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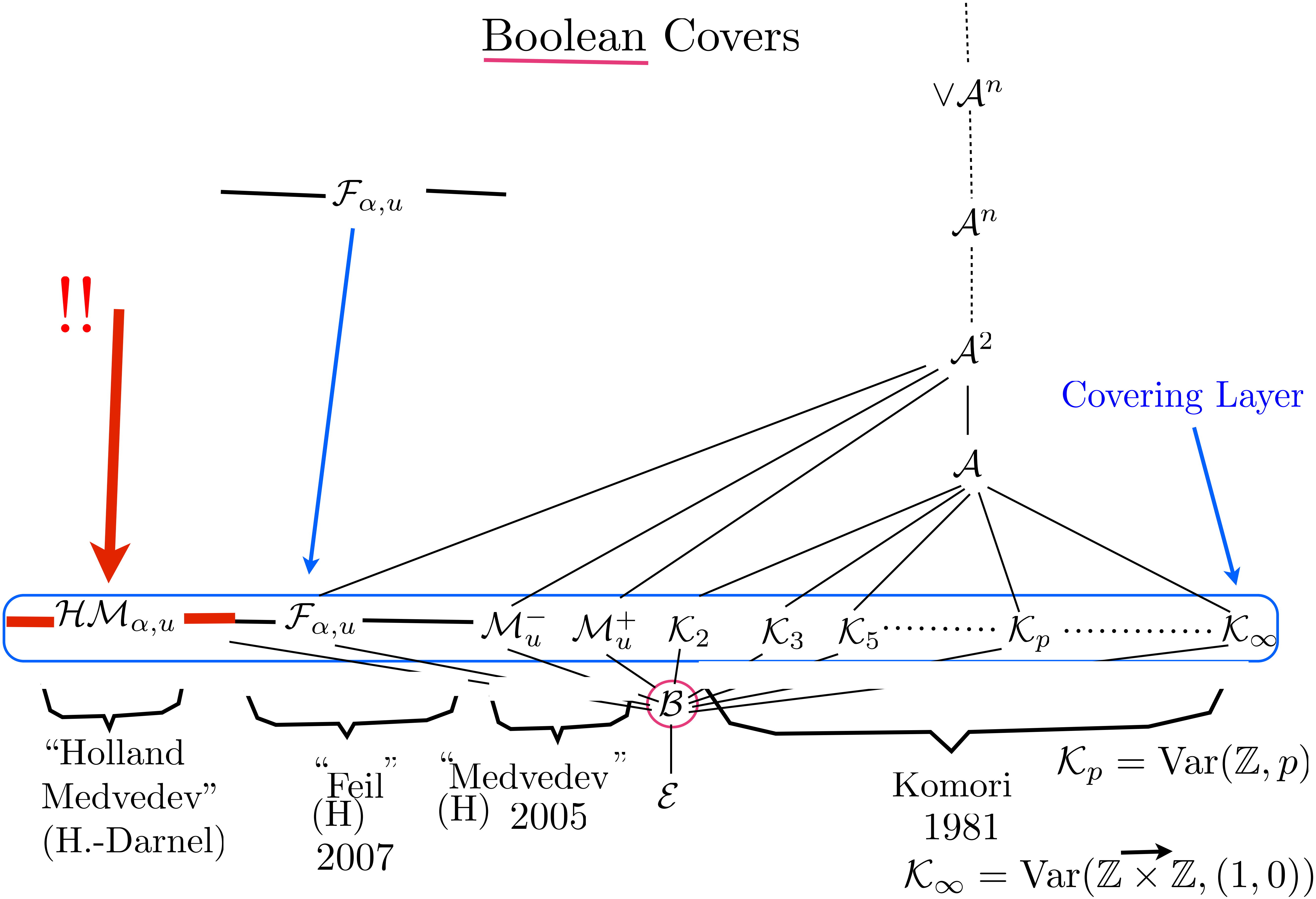

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Theorem.

Let $\mathcal{B}_s$ be the variety generated by (F_s, u). Then $\mathcal{B}_s$ is a cover of the boolean variety $\mathcal{B}$, and if $s \neq t$ then $\mathcal{B}_s \neq \mathcal{B}_t$.

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Therefore, there are uncountably many of these.

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Are there more covers of $\mathcal{B}$?

$* * * * * * * * * * * * * *$

2.

$\mathcal{L}$ = Var(Lattice-Groups) $\quad x = x$

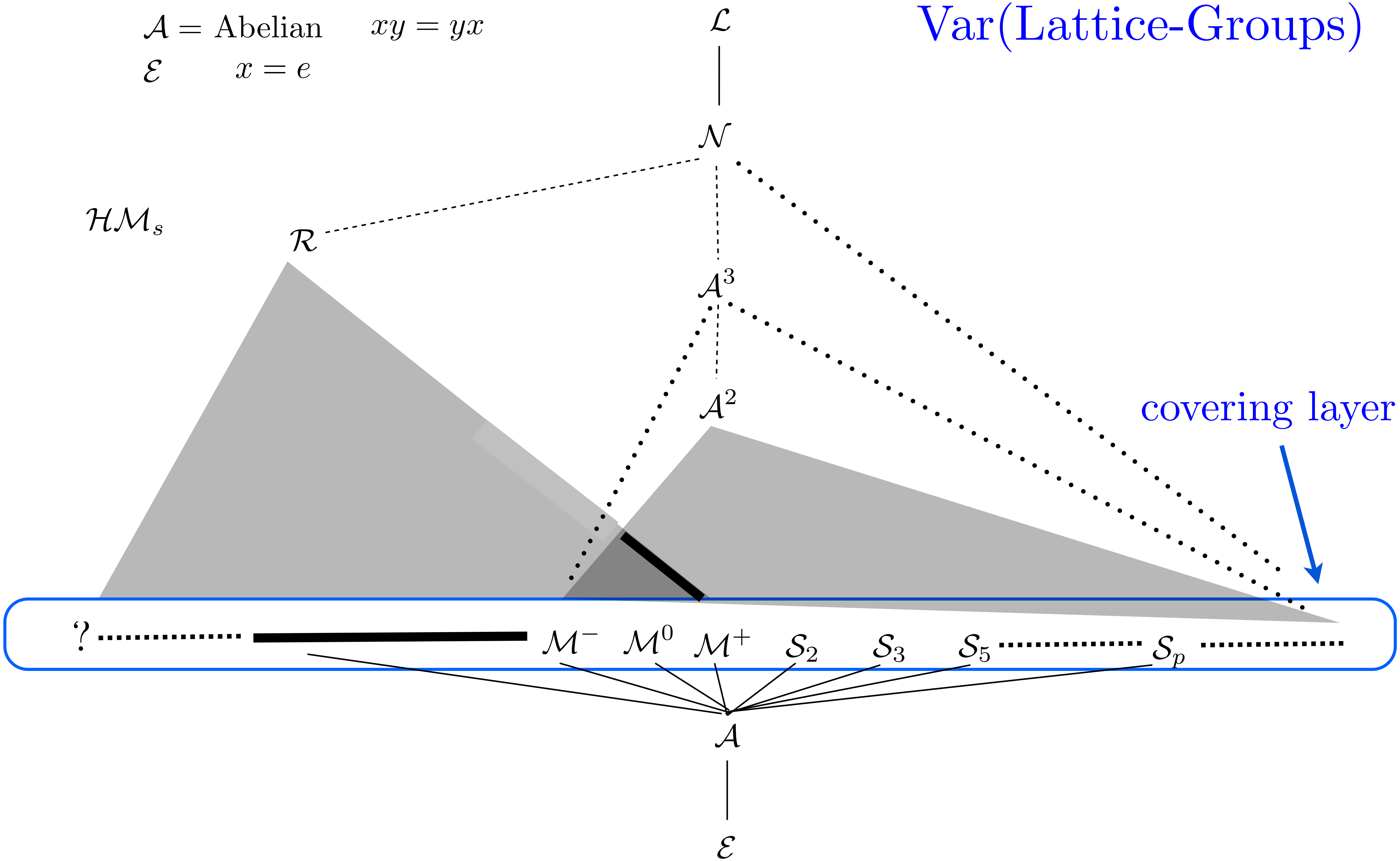
$\mathcal{N}$ = Normal Valued $\quad (x \vee e)(y \vee e) = ((x \vee e)(y \vee e)) \wedge ((y \vee e)^2 (x \vee e)^2)$

$\mathcal{R} \quad (x \vee e) \wedge (z^{-1}(x \wedge e)^{-1} z) = e$ (subdirect product of totally ordered groups)

$\mathcal{A}$ = Abelian $\quad xy = yx$

$\mathcal{E} \quad x = e$

Var(Lattice-Groups)



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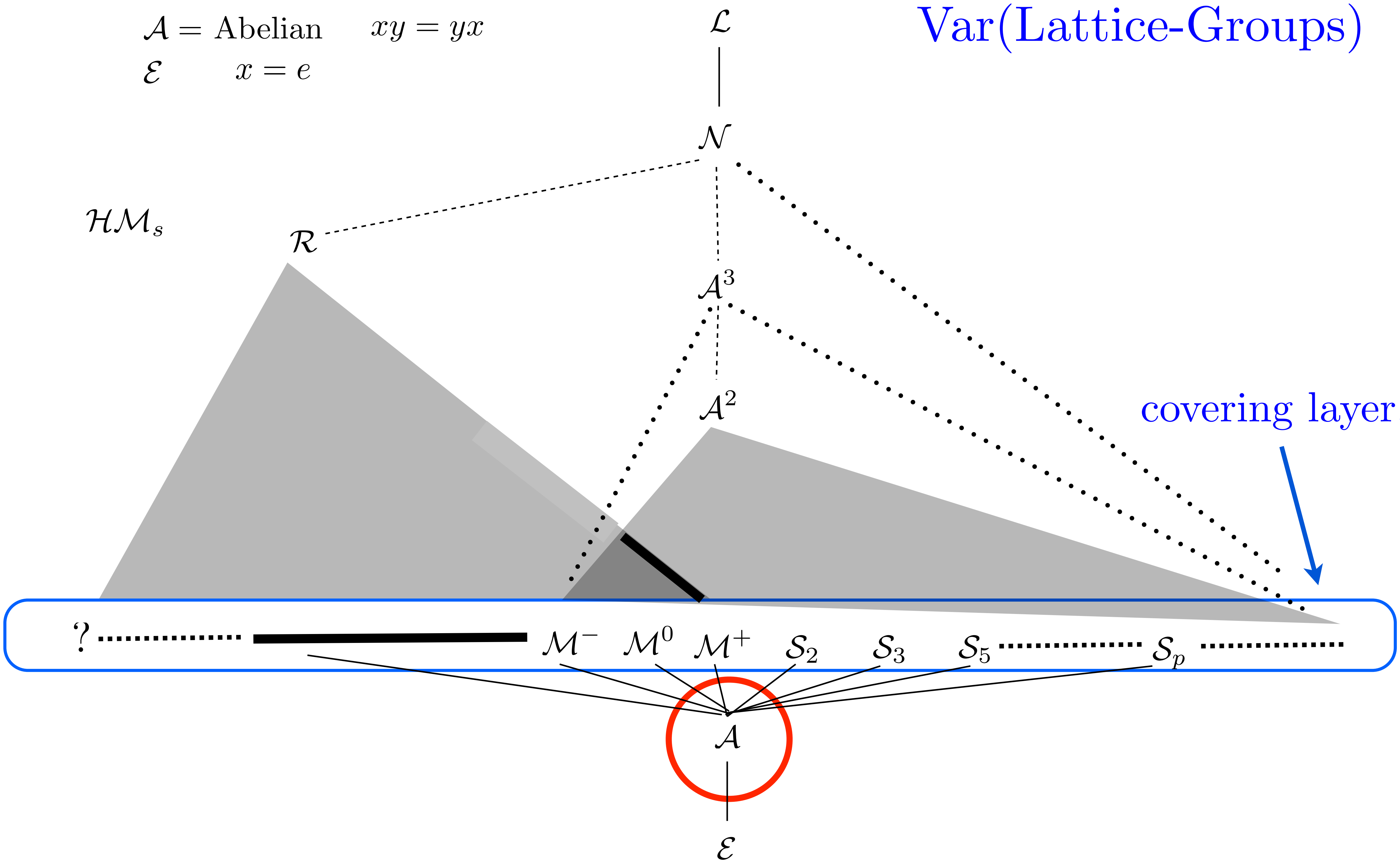

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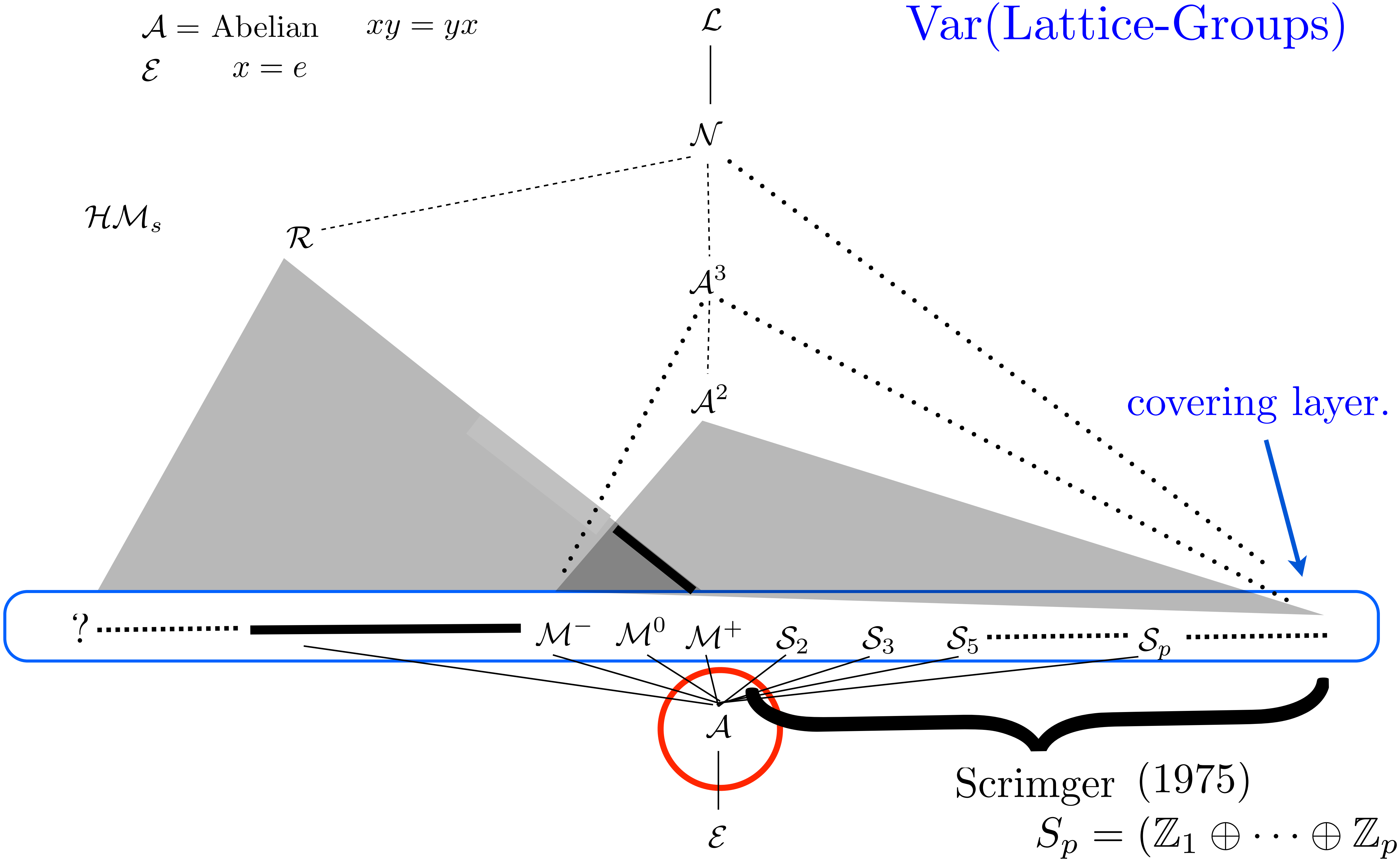

Scrimger (1975)

$$S_p = (\mathbb{Z}_1 \oplus \cdots \oplus \mathbb{Z}_p) \overleftarrow{\rtimes} \mathbb{Z}$$

$p =$ a prime number

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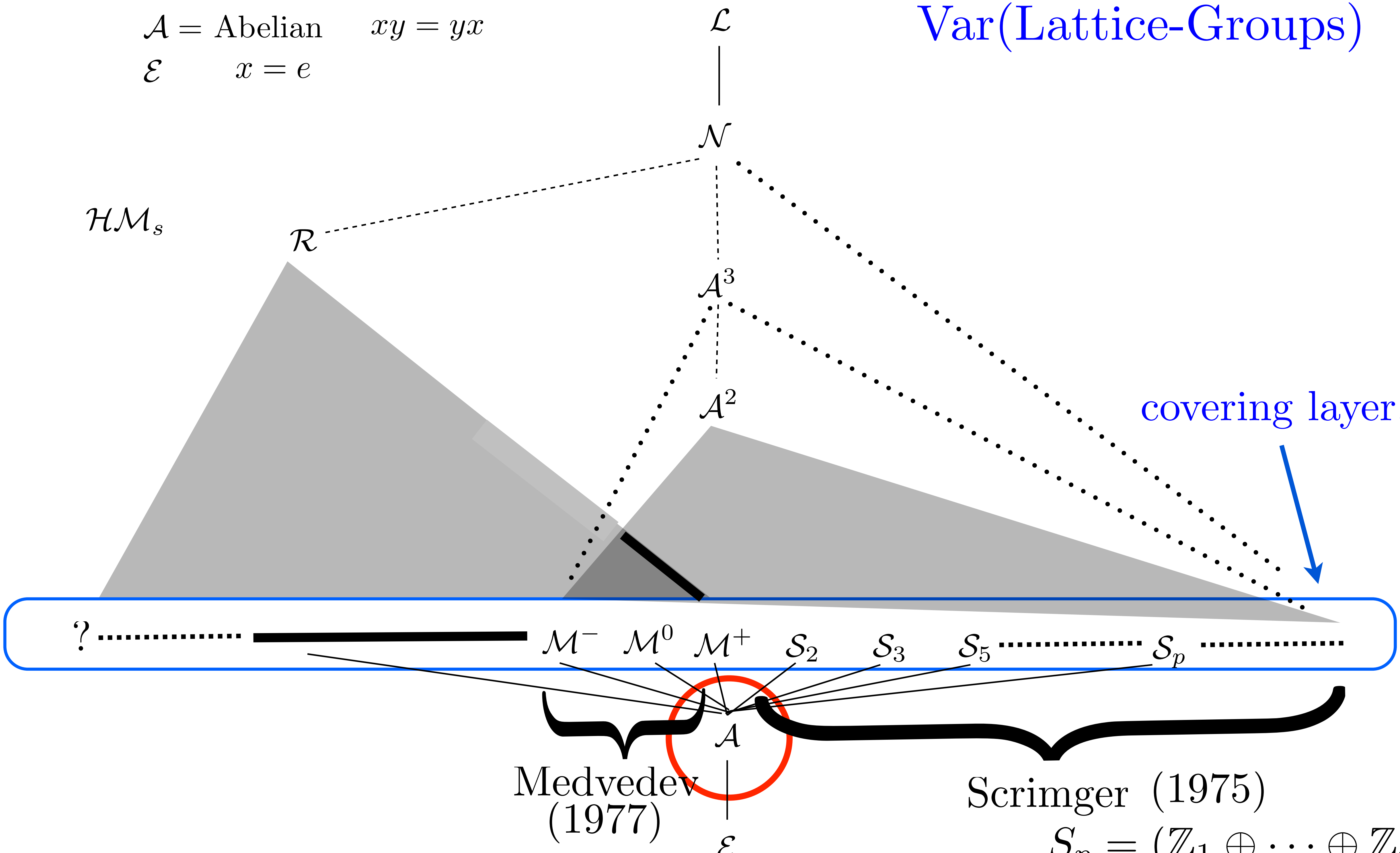

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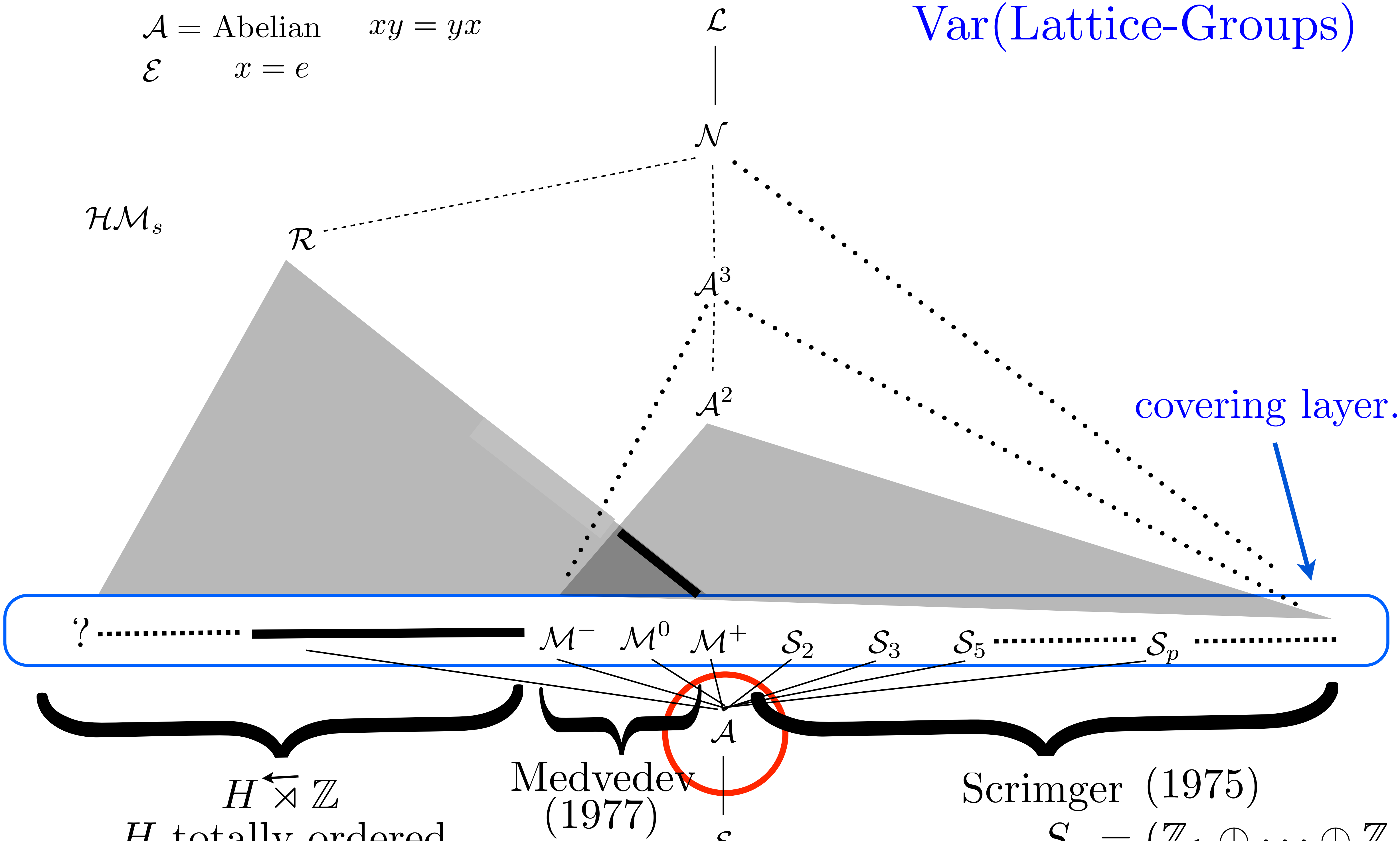

$H \overleftarrow{\rtimes} \mathbb{Z}$

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Scrimger (1975)

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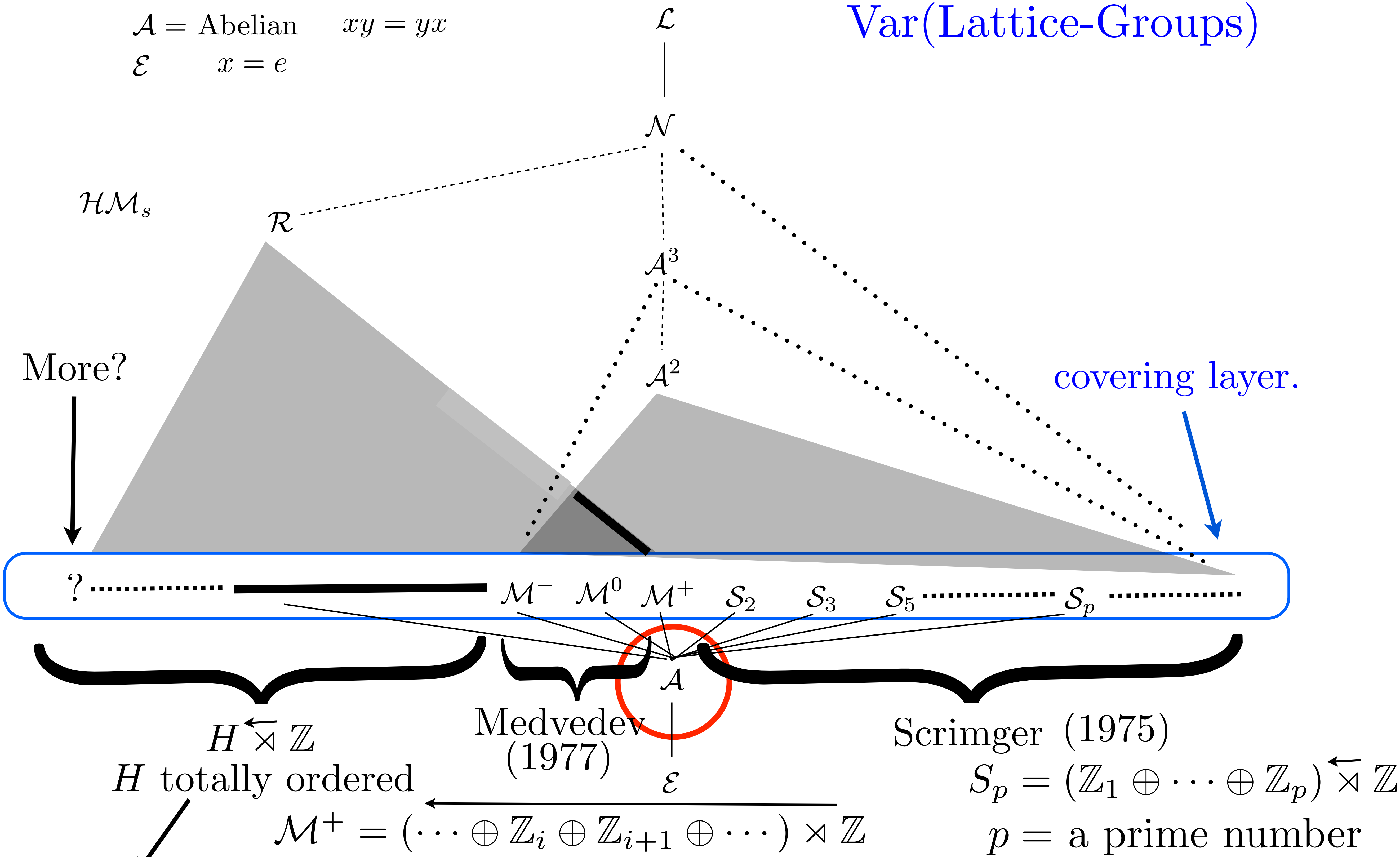

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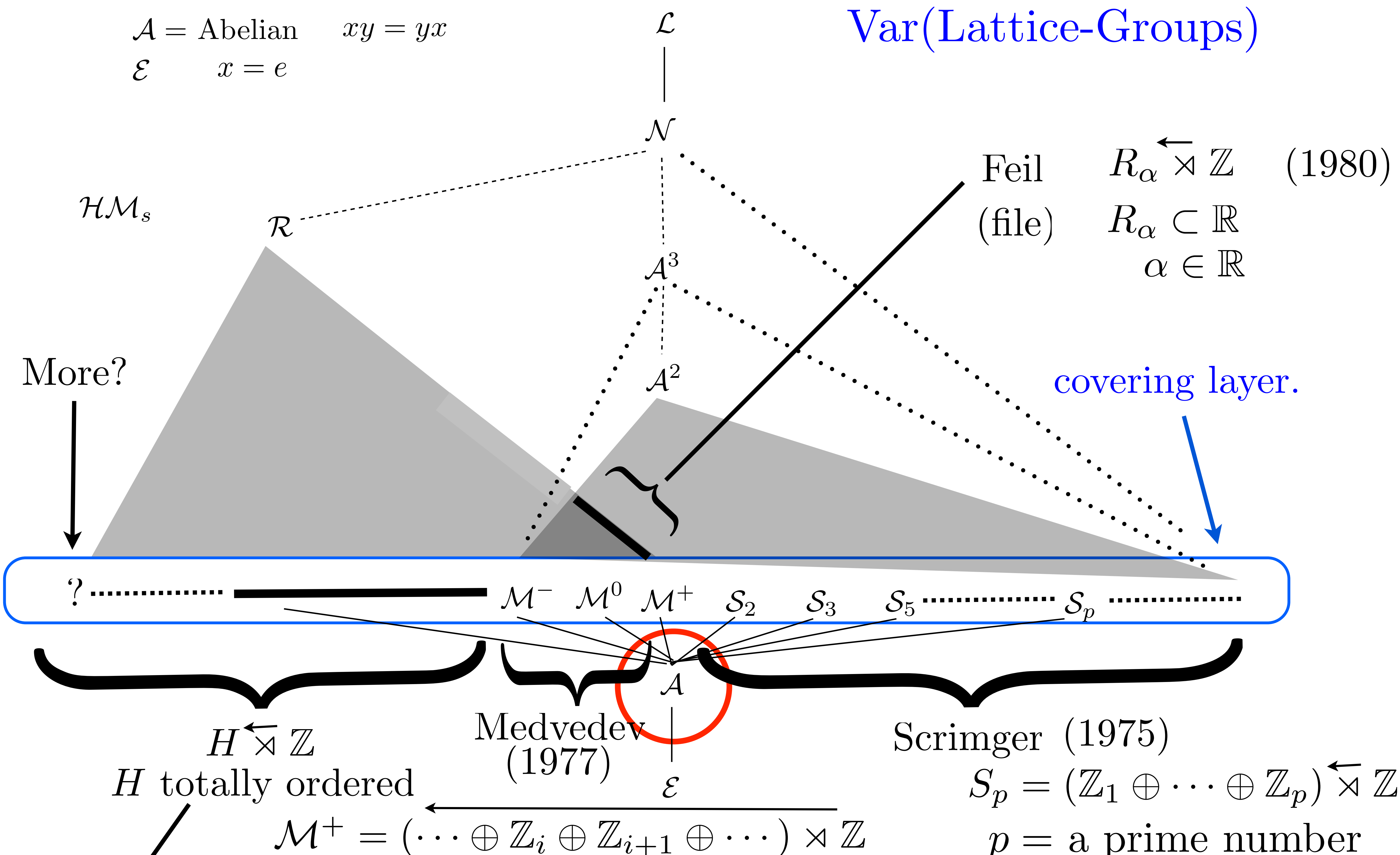

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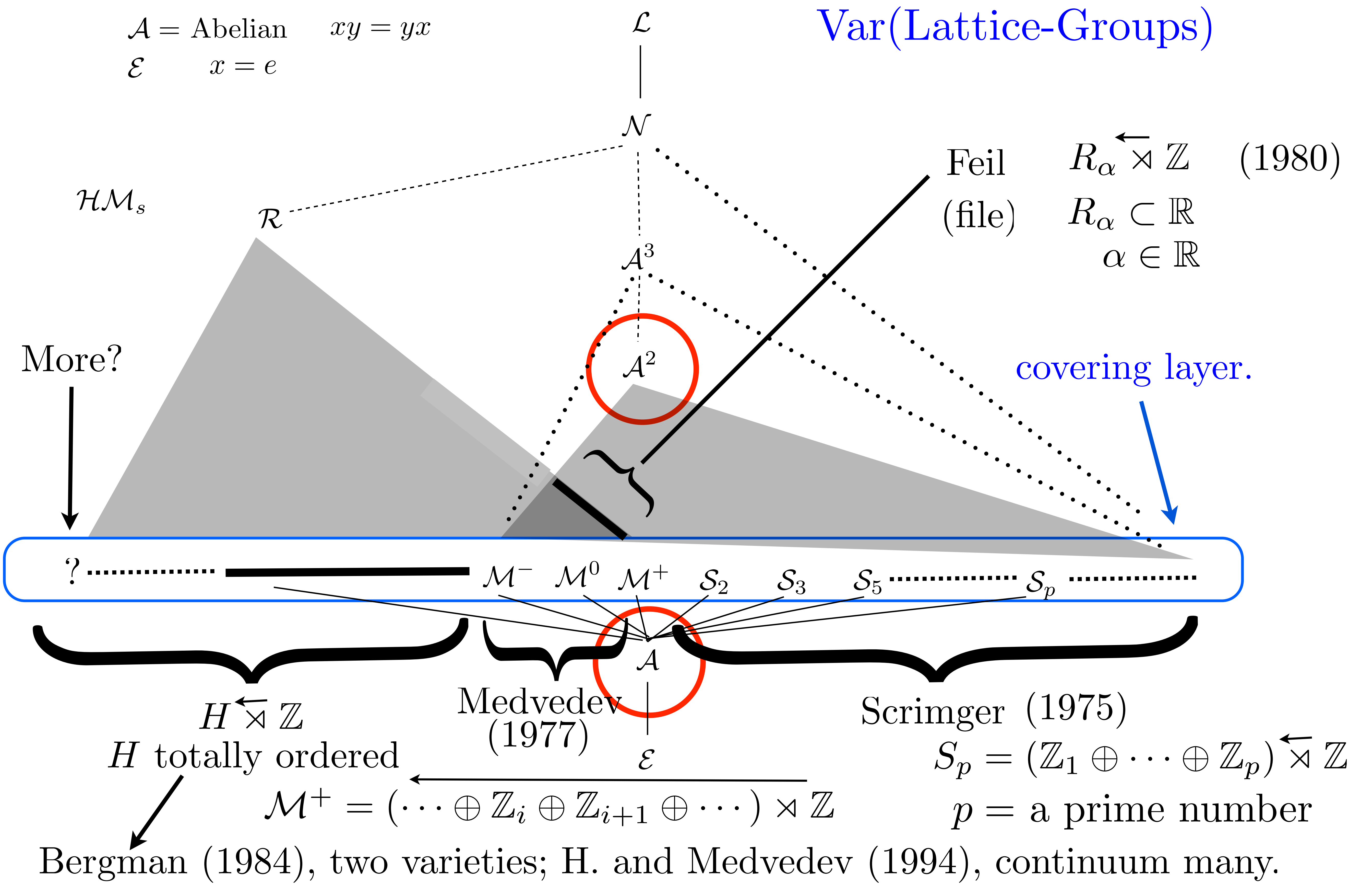

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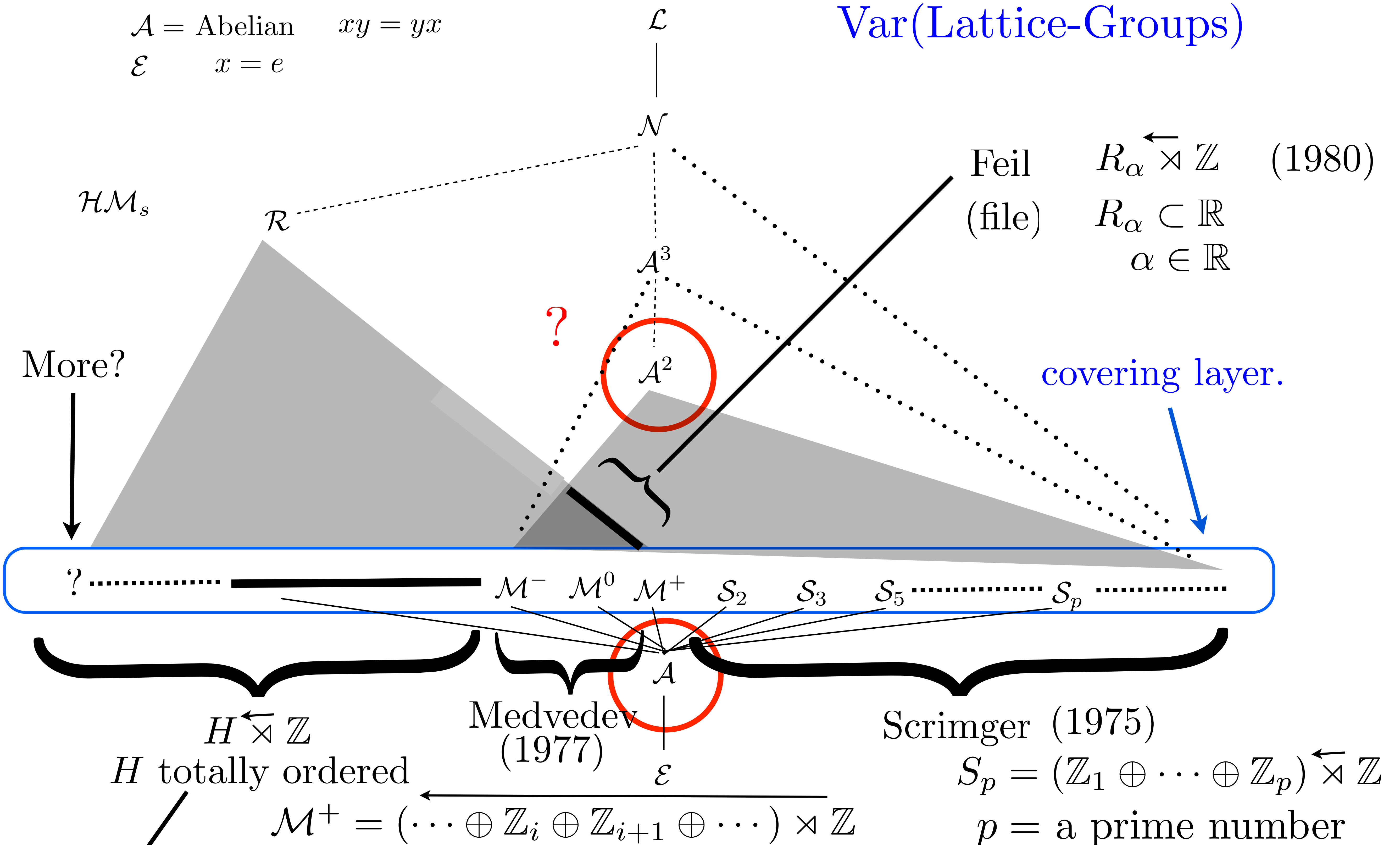

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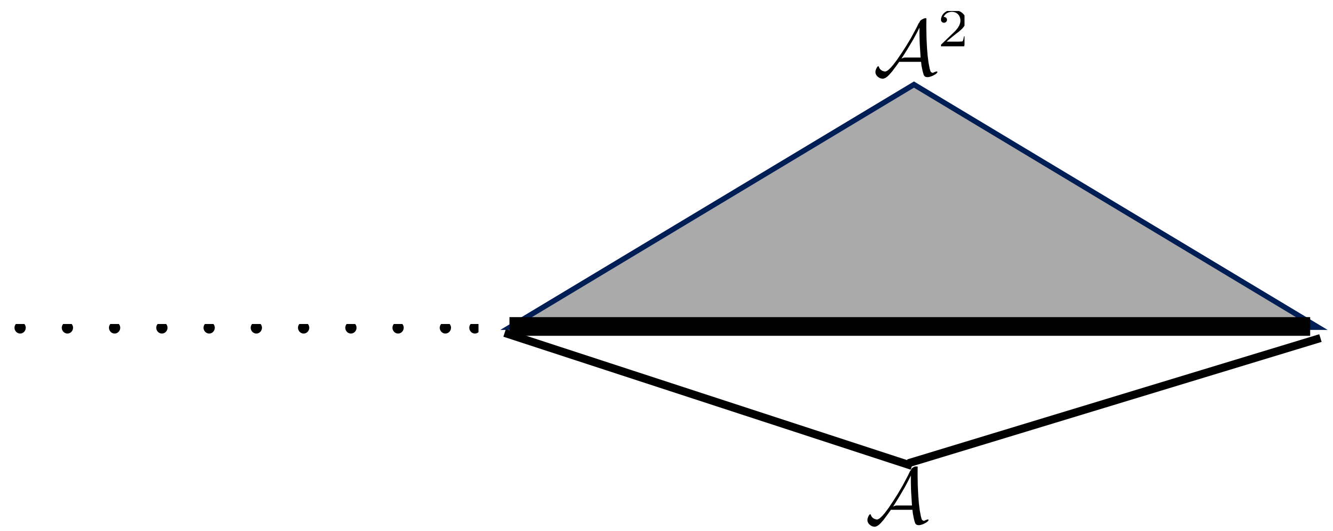

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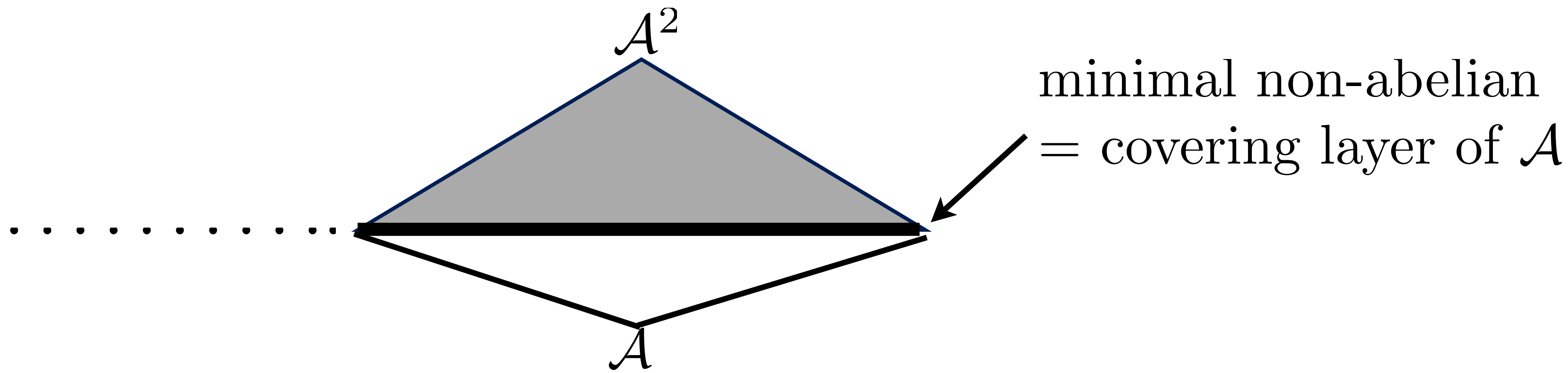

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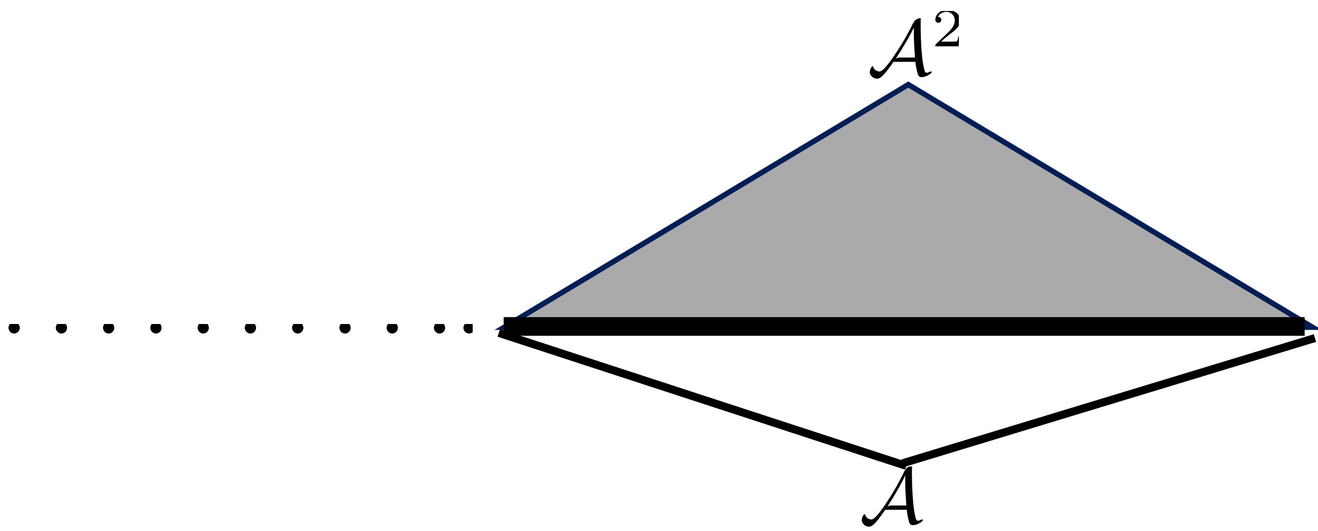

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= covering layer of $\mathcal{A}$

Covering layer of $\mathcal{A}$ under $\mathcal{A}^2$:

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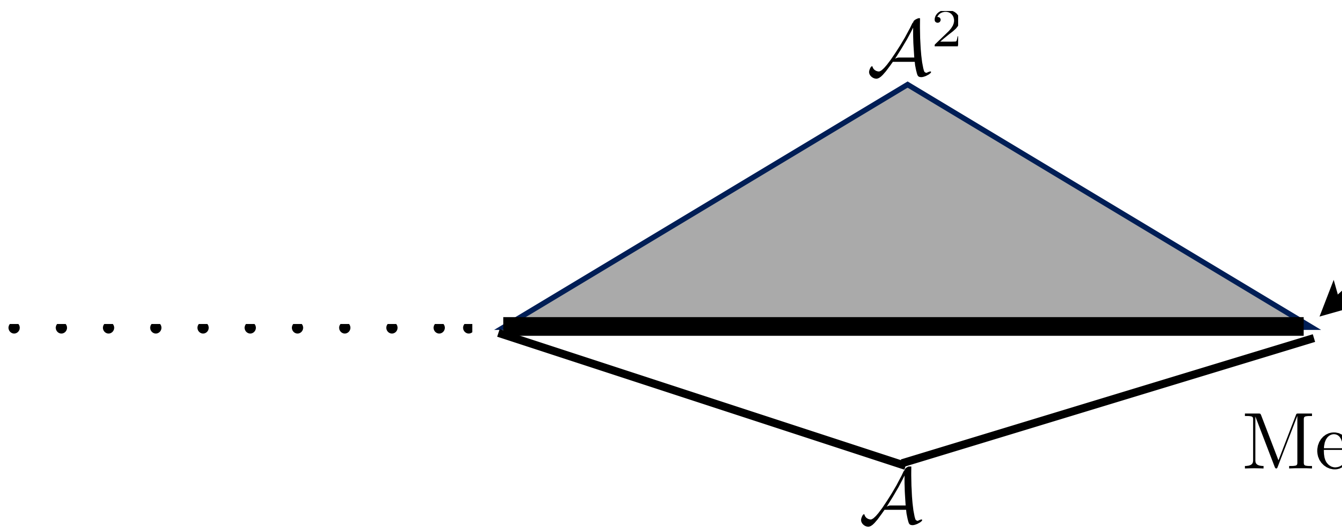

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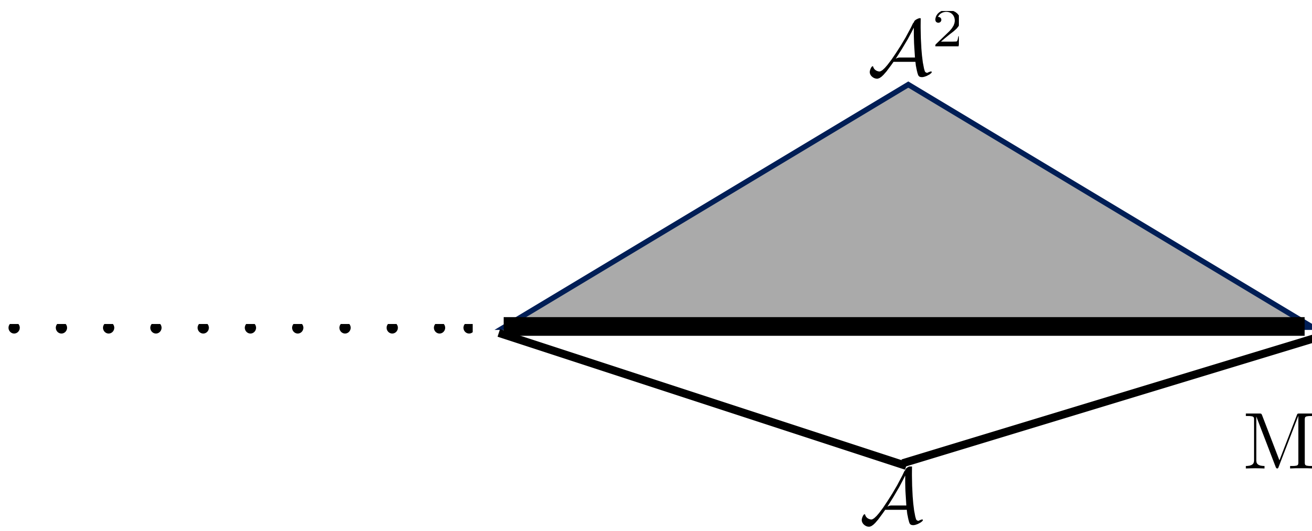

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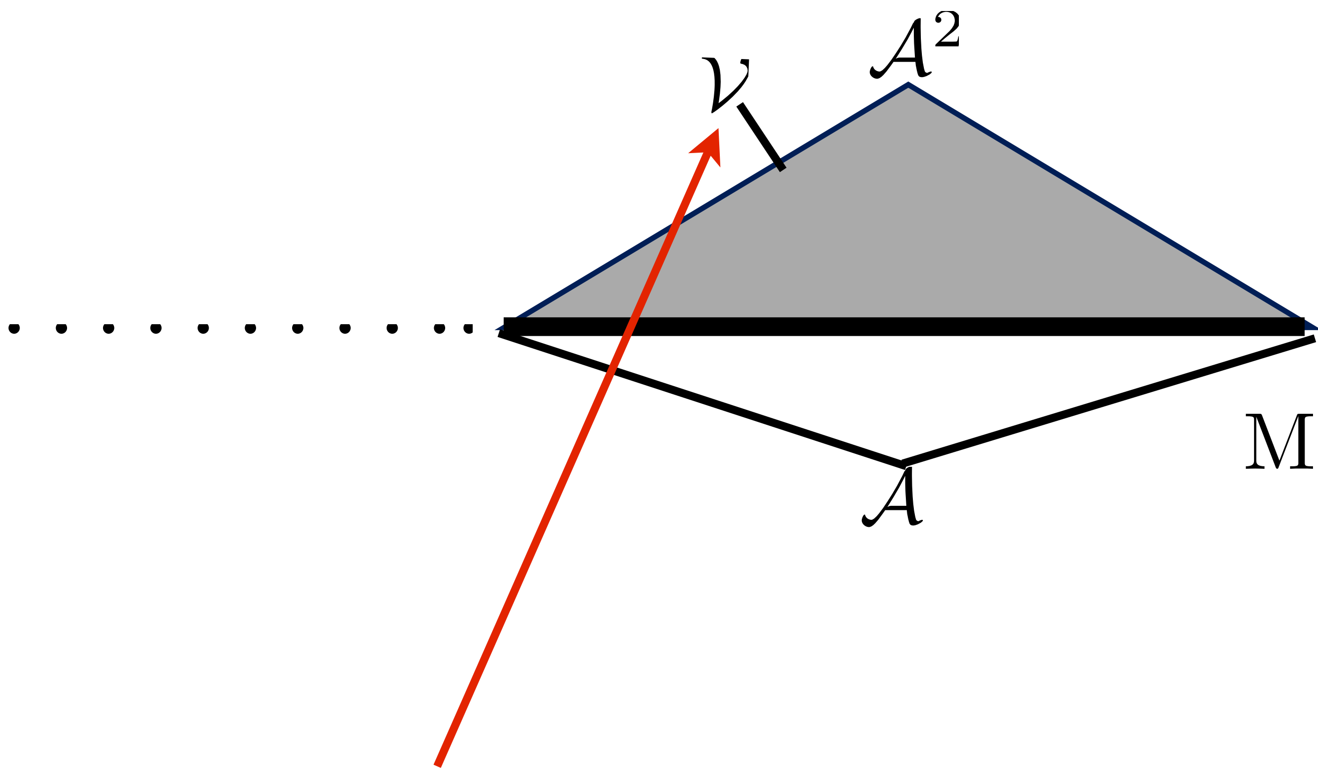

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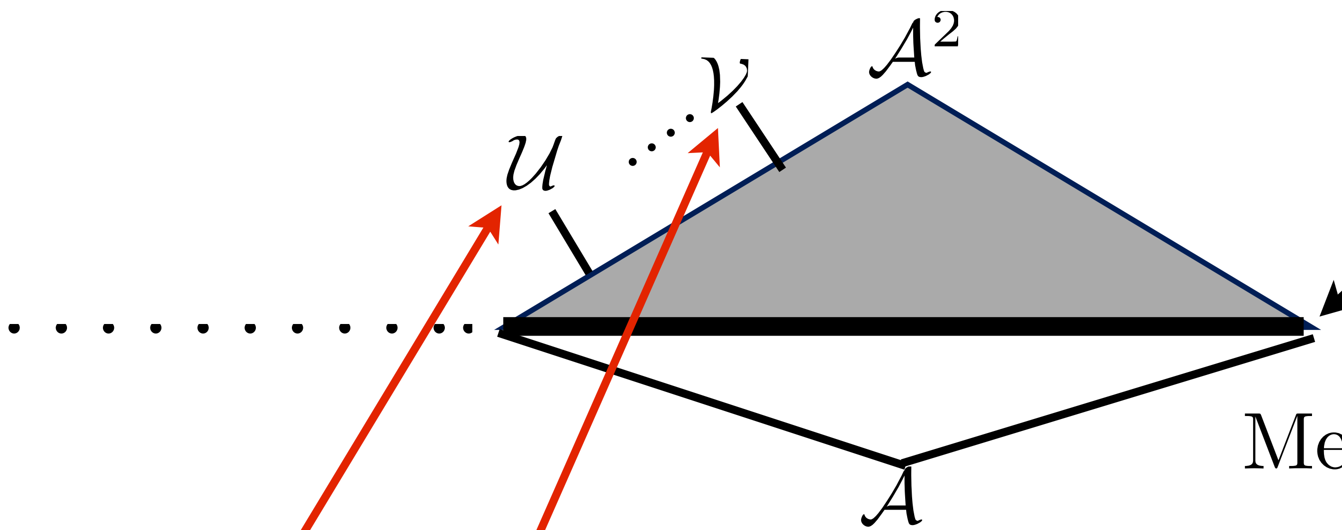

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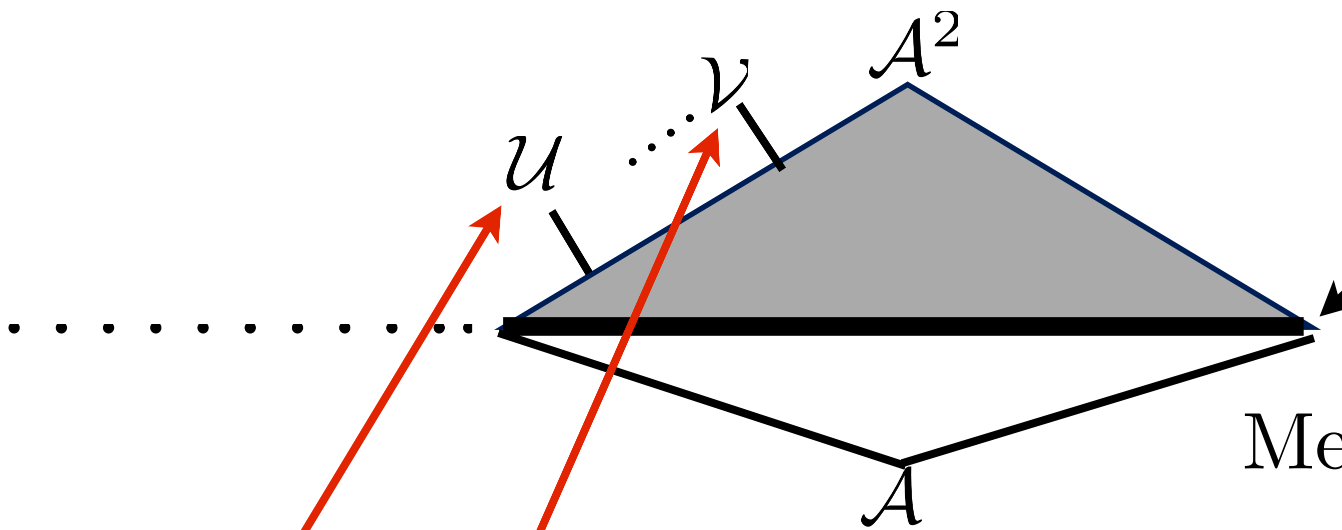

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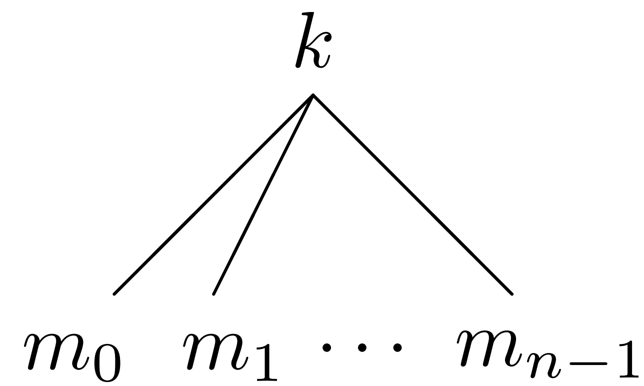

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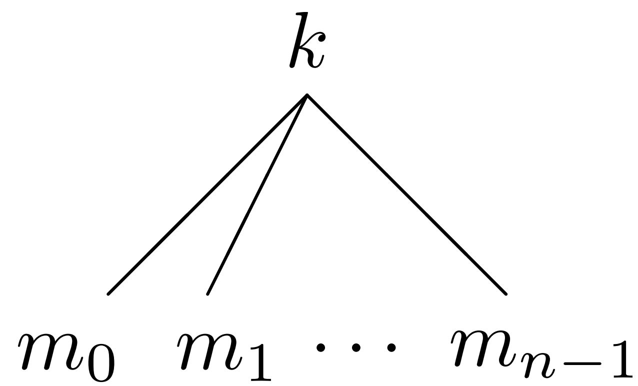

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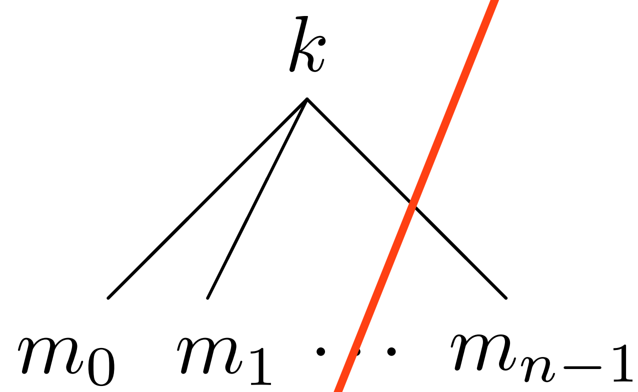

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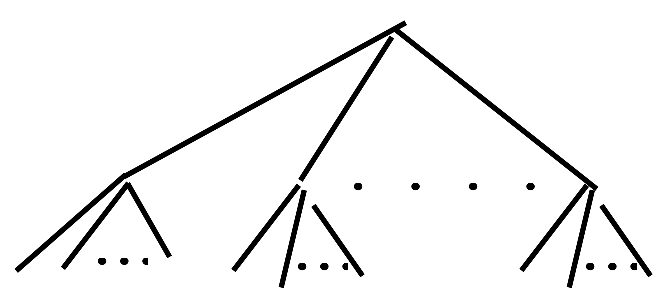

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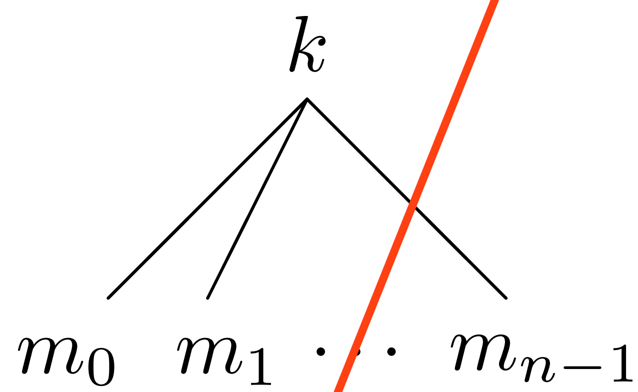

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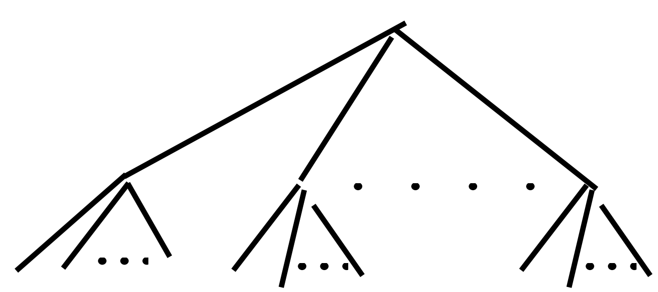

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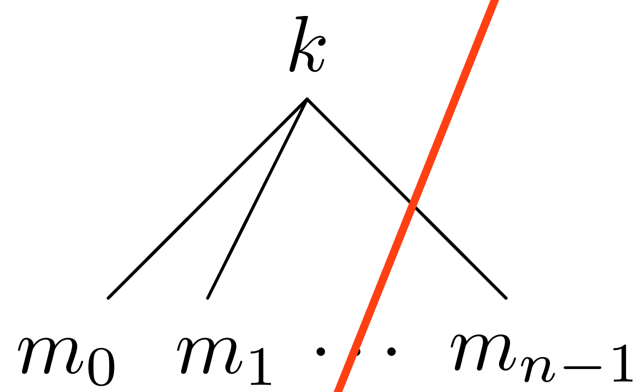

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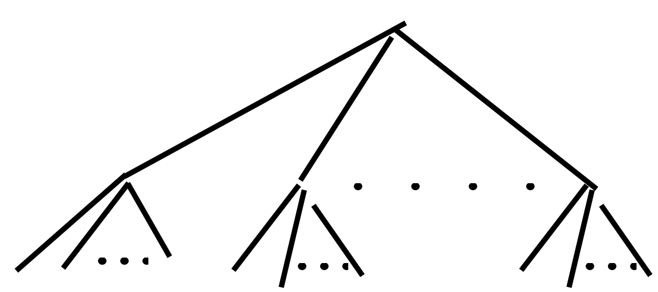

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(D - H)

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The varieties $\mathcal{M}_{p,p,p^k}$ are minimal non-metabelian varieties.

$H_{1,n} = S_n$, a Scrimger ℓ-group of width n and shift by 1.

In general, $H_{r,s}$ is a generalized Scrimger ℓ-group of width rs and shift by r.

$M_{n,r,s}$ is the definition of S_n with H replaced by $H_{r,s}$.

$$M_{n,r,s} = \mathbb{Z}\overrightarrow{\ltimes} \sum_{i=0}^{n-1} H_{r,s}$$

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Some References

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