Graphs admitting a k-NU Polymorphism

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Authors and Abstract

Joint work with

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We characterise, for each $k \geq 3$, the finite undirected graphs that admit a k-ary NU polymorphism, by describing a family generating them by products and retracts.

Graph Basics

- Our graphs are *finite, undirected*, and *without loops*.
- Unless otherwise specified, our graphs are *connected*.
- If **G** is a graph, G is its set of vertices.
- *Product*:
  - vertices of $\mathbf{G} \times \mathbf{H}$: $G \times H$
  - edges of $\mathbf{G} \times \mathbf{H}$: $((g_1, h_1), (g_2, h_2))$ where (g_1, g_2) and (h_1, h_2) are edges of **G** and **H** resp.
- *Retract*:

  R is a retract of **G** if there are edge-preserving maps $e : \mathbf{R} \hookrightarrow \mathbf{G}$ and $r : \mathbf{G} \twoheadrightarrow \mathbf{R}$ such that $r \circ e = id_{\mathbf{R}}$.

  We write $\mathbf{R} \trianglelefteq \mathbf{G}$.

NU Basics

- An operation f on G is a *polymorphism* of the graph **G** if it is edge-preserving, i.e.
  $\forall i\ (x_i, y_i)$ is an edge of **G** $\implies$ $(f(\overline{x}), f(\overline{y}))$ is an edge of **G**.
- For $k \geq 3$, k-ary f is k-NU if

$$f(x, \ldots, x, y, x, \ldots, x) \approx x$$

  for any position of y;
- when $k = 3$, NU operations are called *majority*.

Previous Results and Motivation

We are motivated by the following result:

Theorem (Hell '72 + Bandelt '92 + BL '98 =)

Let **G** *be a graph. TFAE:*

1. **G** *admits a majority polymorphism;*
2. $\mathbf{G} \trianglelefteq \prod_{i=1}^{s} P_i$ *where the* P_i *are paths.*

Analogous results hold for

- posets (Rival $\approx$ '80)
- reflexive graphs (Jawhari, Misane, Pouzet '86)
- but **not** for reflexive digraphs (Kabil, Pouzet '98)

Previous Results and Motivation, continued

Question

Are there/What are the analogs of paths for $k \geq 4$?

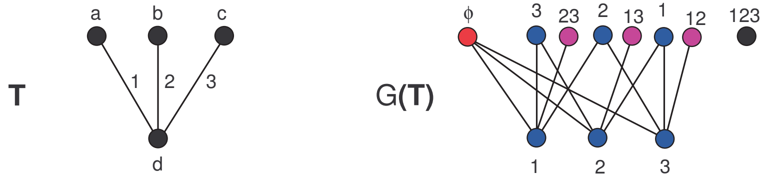
- **Motivation:**
  - $k = 3$ case is natural and cute;
  - NU structures possess remarkable properties from the algorithmic point of view e.g. CSP solvable in NLogspace (Barto, Kozik, Willard '12);
  - for finitely-related structures: NU $\iff$ CD (Barto '13);
- **Obstacle:**
  - proof for $k = 3$: metric properties (absolute retracts) metric approach fails for $k \geq 4$.

The Generating Graphs $\mathbf{G}(\mathbf{T})$

Let $\mathbf{T}$ be a tree with colour classes D and U.
Define a (bipartite) graph $\mathbf{G}(\mathbf{T})$ as follows:

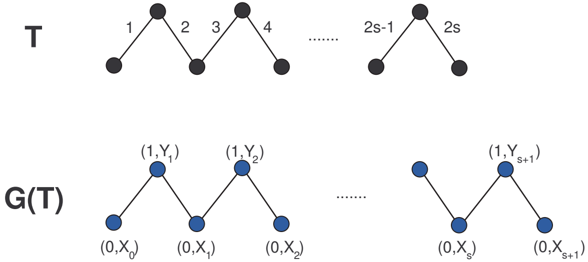
- **Vertices:** there are two kinds:
  - pairs $(0, X)$ where $X \subseteq E(\mathbf{T})$ satisfies $\forall d \in D$ of degree > 1, $\exists! e \in X$ incident to d;
  - pairs $(1, Y)$ where $Y \subseteq E(\mathbf{T})$ satisfies $\forall u \in U$ of degree > 1, $\exists! e \in Y$ incident to u.
- **Edges:** $(0, X)$ and $(1, Y)$ are adjacent if $X \cap Y = \emptyset$.

An Example



The tree **T** and the graph **G**(**T**).
In **G**(**T**)'s diagram, bottom vertices are those of the form $(0, X)$, top ones of the form $(1, Y)$; labels indicate corresponding set of edges, e.g. vertex $(1, \{1, 2\})$ is labelled simply 12.

Another Example



T path of length $s \Rightarrow$ **G**(**T**) path of length $s + 2$
($+$ isolated vertices)

Main Result

Theorem

Let $k \geq 3$ and let $\mathbf{G}$ be a graph. TFAE:

1. $\mathbf{G}$ *admits a k-ary NU polymorphism;*
2. $\mathbf{G} \trianglelefteq \prod_{i=1}^{s} \mathbf{G}(\mathbf{T}_i)$,
   where the T_i are trees with at most $k-1$ leaves.

Sketch of Proof: ($\Leftarrow$)

- k-NU: preserved under products and retracts;
- hence it suffices to prove each $\mathbf{G}(\mathbf{T})$ is k-NU;
- can be built explicitly
  (uses the analogous result for reflexive graphs by Feder, Hell, BL, Loten, Siggers, Tardif)

Finite Duality: Definition

CAUTION: hand-waving ahead.

- A *structure* is a non-empty set together with relations;
- A *homomorphism* between similar structures $=$ relation-preserving map;
  we write $\mathbf{U} \to \mathbf{V}$ if there exists a homomorphism from $\mathbf{U}$ to $\mathbf{V}$ ($\mathbf{U} \not\to \mathbf{V}$ if not).

Definition

A structure $\mathbf{V}$ has *finite duality* if there exist finitely many $\mathbf{T}_1, \dots, \mathbf{T}_s$ such that $\mathbf{U} \not\to \mathbf{V} \iff \exists i\, \mathbf{T}_i \to \mathbf{U}$.

Finite Duality: A few Facts

- The set of "obstructions" $\{\mathbf{T}_1, \ldots, \mathbf{T}_s\}$ is a *duality* for $\mathbf{V}$;
- for each similarity type of structures, $\exists$ notion of *tree*.

Theorem (Nešetřil, Tardif '00, '05)

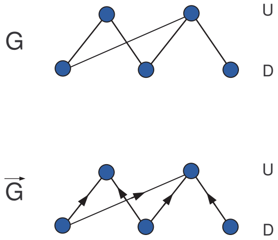
1. *If* $\mathbf{V}$ *has finite duality, then it has a finite duality* $\{\mathbf{T}_1, \ldots, \mathbf{T}_s\}$ *consisting of trees;*
2. *For every tree* $\mathbf{T}$*, there exists a structure* $D(\mathbf{T})$ *such that* $\{\mathbf{T}\}$ *is a duality for* $D(\mathbf{T})$*;*
3. $\{\mathbf{T}_1, \ldots, \mathbf{T}_s\}$ *is a duality for* $\prod_{i=1}^{s} D(\mathbf{T}_i)$*.*

$D(\mathbf{T})$ is called a *dual* of $\mathbf{T}$ (there is an explicit construction.)

Sketch of Proof: (⇒)

Step 1: **G** NU ⇒ **G** bipartite
(BL '98; Bulatov '05 for wnu)

Step 2: **G** has colour classes D and U: let $\overrightarrow{\mathbf{G}}$ denote the *strongly bipartite digraph* obtained from **G** by orienting edges from D towards U:



Sketch of Proof: (⇒), cont'd

Lemma

$\mathbf{G}$ *is* k*-NU* $\iff$ $\overrightarrow{\mathbf{G}}$ *is* k*-NU.*

Step 3: To a digraph $\mathbf{V}$ add all unary relations $\{v\}$, $v \in V$ to obtain a new structure $\mathbf{V}_c$ *with constants*.

Theorem

Let $\mathbf{V}$ *be a strongly bipartite digraph. TFAE:*

1. $\mathbf{V}$ *has an NU polymorphism;*
2. $\mathbf{V}_c$ *has finite duality.*

- uses reduction to posets (BL, Zádori '97)
- FD implies NU for cores (BL, Loten, Tardif '07)

Sketch of Proof: (⇒), cont'd

Step 4: By all the above:

- $\mathbf{G}$ k-NU $\Rightarrow \overrightarrow{\mathbf{G}}_c$ k-NU and has finite duality;
- $\exists$ duality $\{\mathbf{T}_1, \ldots, \mathbf{T}_s\}$ of trees;
- wlog the $\mathbf{T}_i$ are *critical* obstructions;
- critical $\implies$ coloured vertices $=$ the leaves;
- $\overrightarrow{\mathbf{G}}_c$ k-NU $\Rightarrow \forall i$ $\#$ coloured vertices of $\mathbf{T}_i \leq k-1$;
- By def. of dual and duality:

$$\overrightarrow{\mathbf{G}}_c \leftrightarrow \prod_{i=1}^{s} D(\mathbf{T}_i);$$

- since $\overrightarrow{\mathbf{G}}_c$ has constants, it is a core;
- hence $\overrightarrow{\mathbf{G}}_c$ is a retract of the product of the $D(\mathbf{T}_i)$;

Sketch of Proof: (⇒), end

Step 5:

- $\overrightarrow{\mathbf{G}}_c$ is a retract of the product of the $D(\mathbf{T}_i)$;
- "forget" the unary structure and orientation: $\mathbf{G}$ is a retract of the product of the undirected reducts of the duals $D(\mathbf{T}_i)$;
- analysis of the Nešetřil, Tardif construction $+$ etc. : representation in terms of graphs $\mathbf{G}(\mathbf{T})$ only.

Conclusion

Thank you !