MINIMAL CLONES WITH ABELIAN REPRESENTATIONS

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Abstract. We show that a minimal clone has a nontrivial abelian representation if and only if it is isomorphic to a minimal subclone of a finite cyclic group. As an application, we show that a minimal clone contains a Mal'cev operation if and only if it is isomorphic to the clone of idempotent operations of a group of prime order.

1. Introduction

A clone is *trivial* if every operation is projection onto a variable. A clone is *minimal* if it is not trivial, but its only subclone is trivial. Any minimal clone is generated by a single operation. If $\mathcal{C}$ is a clone generated by the operation f, then an *f–representation* of $\mathcal{C}$ is a pair $\mathbf{A} = \langle A; f^{\mathbf{A}} \rangle$ where A is a set, $f^{\mathbf{A}}$ is an operation on A, and the assignment $f \mapsto f^{\mathbf{A}}$ extends to a clone homomorphism from $\mathcal{C}$ to the concrete clone of operations on A generated by $f^{\mathbf{A}}$. Such a representation will be called *trivial* if the clone of $\langle A; f^{\mathbf{A}} \rangle$ is a trivial clone; i.e., if $f^{\mathbf{A}}$ is projection onto a variable. A representation is *faithful* if the assignment $f \mapsto f^{\mathbf{A}}$ extends to a clone isomorphism from $\mathcal{C}$ to the clone of operations on A generated by $f^{\mathbf{A}}$. The *size* of the representation $\langle A; f^{\mathbf{A}} \rangle$ is $|A|$.

Let $\mathcal{C}$ be a minimal clone and let f be an operation generating $\mathcal{C}$. The class of f–representations of $\mathcal{C}$ is a variety of algebras which we denote by $\mathcal{V}$. The clone of $\mathcal{V}$ is $\mathcal{C}$, which is a minimal clone by assumption, and the clone of each $\mathbf{A} \in \mathcal{V}$ is either trivial or it is a minimal clone. If we change from f to a different operation generating $\mathcal{C}$, then we get a term equivalent variety. In this paper we shall often drop the reference to f when we speak of representations and consequently we shall only consider algebras and varieties up to term equivalence.

The *term condition* for an algebra $\mathbf{A}$ (cf. Chapter 3 of [3]) is the assertion that for all operations $t(x, \bar{y})$ in $\mathrm{Clo}(\mathbf{A})$ and all $a, b \in A$, $\bar{u}, \bar{v} \in A^n$ the following implication holds:

$$t(\underline{a}, \bar{u}) = t(\underline{a}, \bar{v}) \Rightarrow t(\underline{b}, \bar{u}) = t(\underline{b}, \bar{v}).$$

An *abelian* algebra is one which satisfies the term condition. When using the term condition in this paper we shall underline the positions where the arguments change in order to make the application more transparent.

Our project is to describe the minimal clones which have nontrivial abelian representations. We approach minimal clones by looking at their nontrivial abelian representations because of an interesting fact about (nonunary) minimal clones. It may be stated as follows: nontrivial abelian representations must be faithful. The classification problem for minimal clones with a nontrivial abelian representation reduces to the classification problem for minimal clones of abelian algebras and this is a problem easily solved.

The description of the minimal subclones of a module follows immediately from our main theorem (which is Theorem 3.11). However, our result is more than this. In the first place, in our arguments we must allow the possibility that some abelian representations are unrelated to modules. But even in the case where a clone $\mathcal{C}$ has a nontrivial representation as a reduct of a module, we must worry about what happens if this representation is not faithful. This paper is entirely concerned with handling these two difficulties. Surprisingly, we find out in Corollary 3.12 that neither difficulty can occur: any minimal clone which has a nontrivial abelian representation must have a faithful representation as a reduct of a finite cyclic group.

Our results enable us to answer a question asked by P. P. Pálfy: Which Mal'cev operations generate minimal clones? Using modular commutator theory, we show that a minimal clone generated by a Mal'cev operation has a nontrivial abelian representation. The answer to Pálfy's question follows immediately from the classification theorem. (In fact, one could arrange a shorter proof of the answer to Pálfy's question by just combining Theorem 3.13 and Lemma 3.9 together with a few arguments from the end of the proof of Lemma 3.4 to make the connection.) Previously, in [8], Á. Szendrei answered Pálfy's question for minimal clones assumed to have a finite faithful representation. Our solution does not require a finiteness assumption.

2. Minimal Clones

Since a minimal clone is generated by any of its operations which is different from a projection, any minimal clone is unary or idempotent. The idempotent minimal clones may be grouped together according to the least arity of an operation which is not a projection. If one skips over the binary idempotent operations which generate minimal clones, one finds that only very special idempotent operations of high arity can generate minimal clones (since every specialization to fewer variables results in a projection). We introduce the following terminology for (some) idempotent operations of arity > 2.

A *Mal'cev* operation is a ternary operation $p(x, y, z)$ such that the equations

$$p(x, y, y) = p(y, y, x) = x$$

hold. A *majority* operation is a ternary operation $M(x, y, z)$ such that the equations

$$M(x, x, y) = M(x, y, x) = M(y, x, x) = x$$

hold. A *minority* operation is a ternary operation $m(x, y, z)$ such that the equations

$$m(x, y, y) = m(y, x, y) = m(y, y, x) = x$$

hold. A *Pixley* operation (sometimes called a 2/3–minority operation) is a ternary operation $P(x, y, z)$ such that the equations

$$P(x, y, y) = P(x, y, x) = P(y, y, x) = x$$

hold. An (*i–th variable*) *semiprojection* is an operation $s(x_1, \dots , x_k)$ of arity ≥ 3 which is not a projection, but whenever two arguments are equal then the value of $s(x_1, \dots , x_k)$ is x_i.

The following theorem describes the five classes of minimal clones. A proof can be found in Chapter 1 of [7].

THEOREM 2.1. *A minimal clone is generated by one of the following types of operations:*

- (I) *a unary operation not equal to a projection;*
- (II) *an idempotent binary operation not equal to a projection;*
- (III) *a majority operation;*
- (IV) *a minority operation; or*
- (V) *a semiprojection.* □

The following is an easy exercise.

THEOREM 2.2. *A clone of class* (I) *is minimal iff it is generated by a unary operation f which is different from the unary projection and for which either $f(f(x)) = f(x)$ holds or else $f^p(x) = x$ for some prime p.* □

Note that if $P(x, y, z)$ is a Pixley operation, then

$$m(x, y, z) := P(P(x, z, y), y, P(y, x, z))$$

is a minority operation. Moreover, this minority operation generates a *proper* subclone of the clone generated by P. To see this, let $\mathcal{C}$

be the clone generated by $m(x,y,z)$. If $P(x,y,z) \in \mathcal{C}$, then every representation of $\mathcal{C}$ would have a Pixley operation: the interpretation of $P(x,y,z)$. Because of the equations that define minority operations and Pixley operations, there is (up to isomorphism) exactly one two–element representation for either type of operation. To prove that $P(x,y,z) \notin \mathcal{C}$ it will suffice to show that a clone on $\{0,1\}$ generated by a minority operation does not include a Pixley operation. On $\{0,1\}$, a minority operation $m(x,y,z)$ must interpret as $x+y+z (\mathrm{mod}\ 2)$, and so $m(x,y,z)$ generates the clone of the two–element affine vector space. Such a representation of $\mathcal{C}$ is abelian. Since

$$P(0,0,\underline{1}) = 1 = P(1,0,\underline{1})$$

and

$$P(0,0,\underline{0}) = 0 \neq 1 = P(1,0,\underline{0}),$$

it follows that $P(x,y,z) \notin \mathcal{C}$. Hence, no Pixley operation generates a minimal clone.

A two–element representation of a minority operation is a nontrivial abelian representation. From the main results of this paper, such a representation is faithful. Hence, the only clone in class (IV), up to isomorphism, is the clone of the two–element affine vector space. This description of the clones in class (IV) was first obtained by I. G. Rosenberg, [6]. We remark that classes (I) and (IV) are the only classes of minimal clones that have been completely described.

3. Abelian Representations

We want to know which classes of minimal clones have nontrivial abelian representations. Every unary algebra is abelian, so all minimal clones of class (I) have nontrivial abelian representations. Some, but not all, of the minimal clones in class (II) have nontrivial abelian representations. The minority equations imply that any clone in class (IV) has a two–element nontrivial abelian representation.

THEOREM 3.1. *No clone in class* (III) *or* (V) *has a nontrivial abelian representation.*

Proof. We show that any abelian representation of a clone which has a majority operation or semiprojection must be trivial. For the majority we have

$$M(x,\underline{x},z) = x = M(x,\underline{x},x).$$

Changing the underlined occurrence of x to y we get

$$M(x,\underline{y},z) \stackrel{?}{=} x = M(x,\underline{y},x).$$

Therefore, if the term condition held, we would have $M(x,y,z) = x$ for all x, y and z. The same argument applied to different variables forces $M(x,y,z) = y$ and $M(x,y,z) = z$ for all x, y and z. But this could only happen in a representation of size one which, according to definitions, is trivial.

If $s(x_1, x_2, x_3, \ldots, x_k)$ is a (first variable) semiprojection, then

$$s(x_1, \underline{x_1}, x_3, \ldots, x_k) = x_1 = s(x_1, \underline{x_1}, x_2, \ldots, x_k).$$

Changing the underlined x_1 to x_2 we get

$$s(x_1, \underline{x_2}, x_3, \ldots, x_k) \stackrel{?}{=} x_1 = s(x_1, \underline{x_2}, x_2, \ldots, x_k).$$

If the term condition held, then $s(x_1, x_2, x_3, \ldots, x_k) = x_1$ would hold for all choices of $(x_1, \ldots, x_k)$ and then s would be first projection. This could not happen in a nontrivial representation. This finishes the proof. $\square$

The only minimal clones with nontrivial abelian representations which we have not yet described are those of class (II). We shall first proceed to describe nontrivial abelian representations of minimal clones generated by an idempotent binary operation, and afterwards we shall describe the clones themselves.

LEMMA 3.2. *If* $\mathbf{A} = \langle A; xy \rangle$ *is an abelian representation of a clone of class* (II)*, then* $\mathbf{A} \models (xy)(zu) = (xz)(yu)$.

Proof. By idempotence we have

$$(\underline{y}y)(z\underline{z}) = yz = (\underline{y}z)(y\underline{z}).$$

Changing the underlined y to x and the underlined z to u gives the result. $\square$

When we refer to clones of class (II), we assume a specifically chosen idempotent binary operation which generates the clone. If $\mathbf{A} = \langle A; xy \rangle$ is a representation of such a clone we will say that the representation $\mathbf{A}$ is *left cancellative* if $\mathbf{A} \models (xy = xz) \Rightarrow y = z$. *Right cancellative* is defined in the obvious way and $\mathbf{A}$ is *cancellative* if it has both properties.

LEMMA 3.3. *Let* $\mathcal{C}$ *be a minimal clone of class* (II) *and assume that* $\mathcal{C}$ *has a nontrivial abelian representation which is neither left nor right cancellative. Then* $\mathcal{C}$ *has a nontrivial abelian representation which is a rectangular band.*

Proof. Let $\mathbf{A}$ be an abelian representation which is neither left nor right cancellative. Since $\mathbf{A}$ is not left cancellative there exist $u, v, w \in A$ such that

$$\underline{u}v = \underline{u}w.$$

Applying the term condition we get that $xv = xw$ for all $x \in A$. In particular, we have $wv = ww = w$ and $v = vv = vw$. Thus, $L = \{v, w\}$ is a subuniverse of $\mathbf{A}$ which supports a subalgebra that is a left zero semigroup. Similarly, the failure of right cancellativity in $\mathbf{A}$ implies that $\mathbf{A}$ has a subalgebra $\mathbf{R}$ which is a two–element right zero semigroup. The algebra $\mathbf{L} \times \mathbf{R}$ is in the variety generated by $\mathbf{A}$, so it is a representation of $\mathcal{C}$. $\mathbf{L} \times \mathbf{R}$ is a rectangular band, thus it is abelian. $\mathbf{L} \times \mathbf{R}$ has a nontrivial clone, so it is a nontrivial abelian representation of $\mathcal{C}$. □

Assume that $\mathcal{C}$ is a minimal clone of class (II) which has a (possibly trivial) abelian representation $\mathbf{A} = \langle A; xy \rangle$ of size > 1 which is not left cancellative. Assume that $\mathcal{C}$ also has a (possibly trivial) abelian representation $\mathbf{B} = \langle B; xy \rangle$ of size > 1 which is not right cancellative. Then $\mathbf{A} \times \mathbf{B}$ is a nontrivial abelian representation that is neither left nor right cancellative. Lemma 3.3 already describes this case in as much detail as we shall require, so we no longer need to consider minimal clones which have an abelian representation of size > 1 which is not left cancellative *and also* an abelian representation of size > 1 which is not right cancellative. We may assume that $\mathcal{C}$ is a minimal clone of class (II) for which every abelian representation is right cancellative.

LEMMA 3.4. *Let $\mathcal{C}$ be a minimal clone of class* (II) *and assume that xy is right cancellative in every abelian representation of $\mathcal{C}$. Assume that $\mathcal{C}$ has no nontrivial abelian representation satisfying the equation $x(xy) = x$. Then $\mathcal{C}$ has a nontrivial abelian representation as a vector space over a prime field.*

Proof. We shall use the following notation: for a representation $\mathbf{A}$ and for $a \in A$ we write $\lambda_a(x)$ for the polynomial $x \mapsto ax$. We write $\rho_a(x)$ for the polynomial $x \mapsto xa$. When $\mathbf{A}$ is abelian, Lemma 3.2 implies that

$$\lambda_a(x)\lambda_a(y) = (ax)(ay) = (aa)(xy) = a(xy) = \lambda_a(xy),$$

so each λ_a (and each ρ_a) is an endomorphism. The term condition implies that the kernel of λ_a is independent of a ($\underline{a}u = \underline{a}v \Leftrightarrow \underline{b}u = \underline{b}v$), and a similarly for ρ_a. The same argument shows that $\lambda_a \circ \lambda_b(x)$ has kernel which is independent of both a and b, etc. Our hypothesis that every abelian representation of $\mathcal{C}$ is right cancellative implies that each

ρ_a has trivial kernel. Assuming that the equation $x(xy) = x$ fails is the statement that $\lambda_x \circ \lambda_x$ is not constant when $\langle A; xy\rangle$ is a nontrivial abelian representation.

Let $\mathbf{A}$ be a nontrivial abelian representation of $\mathcal{C}$. Since $\mathbf{A}$ is a right cancellative, idempotent, abelian groupoid; Theorem 2.11 of [4] proves that the relation $\ker(\lambda_x) = \theta = \{(a, b) \in A^2 \mid aa = ab\}$ is a congruence on $\mathbf{A}$ and that $\mathbf{A}/\theta$ is a subalgebra of a reduct of a (unital) module. In particular, $\mathbf{A}/\theta$ is another abelian representation of $\mathcal{C}$. If $\mathbf{A}/\theta$ is trivial as a representation of $\mathcal{C}$; then, since every abelian representation of $\mathcal{C}$ is right cancellative, it must be that $\mathbf{A}/\theta$ is a left zero semigroup. Therefore, if $\mathbf{A}/\theta$ is a trivial abelian representation of $\mathcal{C}$, we have $\mathbf{A}/\theta \models xy = x$ and so $\mathbf{A} \models xy\ \theta\ x$. From the definition of θ we get that $\mathbf{A} \models x(xy) = xx = x$. But we have assumed that no nontrivial abelian representation of $\mathbf{A}$ satisfies the equation $x(xy) = x$, so we conclude that $\mathbf{A}/\theta$ is a nontrivial representation of $\mathcal{C}$ which is a subalgebra of a reduct of a module.

Changing notation, we assume that our original choice of $\mathbf{A}$ was a subalgebra of a reduct of a module. In fact, as argued in Section 3 of [4], the variety generated by $\mathbf{A}$ contains an algebra $\mathbf{A}'$ which extends $\mathbf{A}$, generates the same variety as $\mathbf{A}$, and is actually a reduct of a module. Therefore, we may (and do) assume that $\mathbf{A}$ is a reduct of a module ${}_{\mathbf{R}}M$. One can represent the basic operation of $\mathbf{A}$ as an idempotent module operation: $xy = (1-r)x + ry$ with $r \in R$. Now, for $\mathbf{R}'$ equal to the subring of the additive endomorphism ring of ${}_{\mathbf{R}}M$ generated by the function $x \mapsto r \cdot x$, $\mathbf{A}$ is a subalgebra of a reduct of the $\mathbf{R}'$–module structure on ${}_{\mathbf{R}}M$. Hence we may assume that $\mathbf{R}$ is a subring of additive endomorphisms of ${}_{\mathbf{R}}M$ and that $\mathbf{R}$ is generated as a ring by the single element r. In particular, this implies that $\mathbf{R}$ is a commutative ring which acts faithfully on ${}_{\mathbf{R}}M$.

Claim. $\langle R; (1-r)x + ry\rangle$ is a nontrivial abelian representation of $\mathcal{C}$.

Proof of Claim. If ${}_{\mathbf{R}}N$ is a submodule of ${}_{\mathbf{R}}M^{\kappa}$ and θ is a congruence on ${}_{\mathbf{R}}N$, then the reduct ${}_{\mathbf{R}}N'$ of ${}_{\mathbf{R}}N$ to the operation $(1-r)x + ry$ is a subalgebra of $\mathbf{A}^{\kappa}$ which has θ as a congruence. The algebra ${}_{\mathbf{R}}N'/\theta$ is the reduct of the module ${}_{\mathbf{R}}N/\theta$ to the operation $(1-r)x + ry$. Therefore, the variety generated by $\mathbf{A}$ contains all reducts to $(1-r)x + ry$ of modules in $\mathsf{HSP}({}_{\mathbf{R}}M)$. Since ${}_{\mathbf{R}}M$ is a faithful $\mathbf{R}$–module, this implies that ${}_{\mathbf{R}}R \in \mathsf{HSP}({}_{\mathbf{R}}M)$. Therefore, $\langle R; (1-r)x + ry\rangle$ is a representation of $\mathcal{C}$. If it is a trivial representation, then $(1-r)x + ry$ is a projection operation which forces $r = 0$ or 1. But this would force $\langle M; (1-r)x + ry\rangle$ to be a trivial representation also, and it is not. Hence, the claim is proved.

Now we may assume that $\mathbf{A} = \langle R; (1-r)x+ry\rangle$ where $r \neq 0, 1$. We consider representations of $\mathcal{C}$ constructed from ideals of $\mathbf{R}$ as follows: if I is a proper ideal of $\mathbf{R}$, then

$$\theta := \{(a, b) \in R^2 \mid a - b \in I\}$$

is a congruence on the module ${}_{\mathbf{R}}R$, hence on the reduct $\mathbf{A} = \langle R; (1-r)x+ry\rangle$. The quotient $\mathbf{A}/\theta$ has more than one element and is a reduct of ${}_{\mathbf{R}}R/I$. Thus, $\mathbf{A}/\theta$ is an abelian representation of $\mathcal{C}$ which has more than one element. It is easy to check that the following statements are true:

(i) Right cancellativity of $\mathbf{A}/\theta$ is equivalent to the implication

$$(1-r)u \in I \Rightarrow u \in I.$$

(ii) Left cancellativity of $\mathbf{A}/\theta$ is equivalent to the implication

$$ru \in I \Rightarrow u \in I.$$

(iii) The element $1-r$ is a unit. (This follows from the right cancellativity of $\mathbf{A}/\theta$. If $1-r$ was not a unit, then item (i) would fail for $I = (1-r)R$ and $u = 1$.)

(iv) $\mathbf{A}/\theta$ is a trivial representation iff $r \in I$.

(v) $\mathbf{A}/\theta$ satisfies $x(xy) = x$ iff $r^2 \in I$.

If r is in the Jacobson radical of $\mathbf{R}$ and $rR = r^2R$, then $rR = 0$ by Nakayama's Lemma (Proposition 2.6 of [1].) But this would imply $\mathbf{A} \models xy = x$, which is false since $\mathbf{A}$ is a nontrivial representation. Hence, $r \notin r^2R$ if r is in the Jacobson radical. Taking $I = r^2R$ in this case, items (iv) and (v) prove that $\mathbf{A}/\theta$ is a nontrivial abelian representation of $\mathcal{C}$ which satisfies $x(xy) = x$. None exists, by the hypotheses of this lemma, so we conclude that r does not belong to the Jacobson radical.

Since r is not in the Jacobson radical, $\mathbf{R}$ has a maximal ideal K which does not contain r. Choosing $I = K$ in the construction of $\mathbf{A}/\theta$, we get (by item (iv)) that $\mathbf{A}/\theta$ is a nontrivial abelian representation of $\mathcal{C}$. The algebra $\mathbf{A}/\theta$ must be both right and left cancellative. (Right cancellativity follows from items (i) and (iii). For left cancellativity we apply item (ii): if $ru \in K$, then $L = \{s \in R \mid su \in K\}$ is an ideal containing $K \cup \{r\}$. Since K is maximal and $r \notin K$, we get $L = R$. Hence, $1 \in L$ which means $u = 1u \in K$.)

The ring $\mathbf{F} = \mathbf{R}/K$ is a one–generated field, and therefore a finite field. $\mathbf{A}/\theta$ is a reduct of the $\mathbf{F}$–space, ${}_{\mathbf{F}}F$. From the previous paragraph, the operation xy interprets in $\mathbf{A}/\theta$ as $(1-s)x + sy$ for some $s \in F - \{0, 1\}$. The algebra $\mathbf{A}/\theta$ satisfies the equations $\lambda_y^k(x) = x = \rho_y^k(x)$ for $k = |F| - 1$, so the operation xy is invertible in both variables. This implies that $\mathbf{A}/\theta$ has division quasigroup polynomials and therefore a Mal'cev operation. This Mal'cev operation must be $x-y+z$, which is the unique Mal'cev operation of the vector space ${}_{\mathbf{F}}F$. The idempotent binary operations of $\mathbf{A}/\theta$ are represented by

$$\{(1-t)x+ty \mid t \in F'\}$$

where $\mathbf{F}'$ is a subfield of $\mathbf{F}$. But this subfield must be all of $\mathbf{F}$, since $\mathbf{F}$ is a homomorphic image of $\mathbf{R}$, $\mathbf{R}$ is generated by r, and $r+K \in F'$. This implies that $\mathbf{A}/\theta$ is an affine $\mathbf{F}$–space. Now it is clear that $\mathbf{F}$ must be a prime field, since if $\mathbf{F}$ had a proper subfield $\mathbf{L} < \mathbf{F}$, then the subclone of $\mathbf{A}/\theta$ generated by $x-y+z$, which equals all of $\mathrm{Clo}(\mathbf{A})$ by minimality, would contain no operation $(1-u)x+uy$ with $u \in F-L$. $\square$

The remaining case to consider is when every abelian representation of $\mathcal{C}$ is right cancellative and there is at least one nontrivial abelian representation which satisfies $x(xy)=x$.

LEMMA 3.5. *Let $\mathcal{C}$ be a minimal clone of class* (II) *where every abelian representation is right cancellative. Assume that* $\mathbf{C}$ *has at least one nontrivial abelian representation which satisfies $x(xy)=x$. Then for some prime p the algebra*

$$\mathbf{P} = \langle \{0,\dots,p^2-1\}\ ;\ ((1-p)x+py)\pmod{p^2}\rangle$$

is a nontrivial abelian representation of $\mathcal{C}$. (Furthermore, for all primes p the clone of $\mathbf{P}$ *is a minimal clone.)*

Proof. Let $\mathbf{A}$ be a nontrivial abelian representation of $\mathcal{C}$ which satisfies $x(xy)=x$. Since $\mathbf{A}$ is a nontrivial representation, there exist $u,v \in A$ such that $uv \neq u$. The subalgebra generated by $\{u,v\}$ is abelian and still affords a nontrivial representation of $\mathcal{C}$, since otherwise we must have $uv=v=vv$ and this contradicts the right cancellativity of abelian representations. Therefore, we assume that $\{u,v\}$ generates $\mathbf{A}$.

The term condition applied to

$$\underline{x}(xy) = x = \underline{x}(xx)$$

yields

$$\underline{z}(xy) = zx = \underline{z}(xx).$$

Since $\mathbf{A} \models x(xy)=x$, we get $\mathbf{A} \models z(xy)=zx$. It follows from this that any operation of $\mathbf{A}$ agrees with a left–associated product of variables. Furthermore, Lemma 3.2 shows that $(xy)(zu)=(xz)(yu)$ holds. Applying the equation $z(xy)=zx$ to this we get

$$(xy)z = (xy)(zu) = (xz)(yu) = (xz)y.$$

The equation $(xy)z = (xz)y$ implies that any two variables different from the left–most variable may be interchanged without affecting the product. Thus any term operation $t^{\mathbf{A}}(x_1, \dots, x_k)$ agrees on A with one of the form

$$x_i x_1^{n_1} \cdots x_{i-1}^{n_{i-1}} x_{i+1}^{n_{i+1}} \cdots x_k^{n_k}$$

where this product is left–associated and variables are distinct. (Multiple occurrences of x_i could be moved to the right of the left–most x_i and then absorbed with the idempotent law.)

Claim. There is a prime p such that $uv^p = u$.

Proof of Claim. Assume that the claim is false. Since $\mathbf{A}$ is right cancellative, the function $\rho_v(x)$ is a 1-1 function. Either the sequence $\Sigma = (u, uv, uv^2, \dots)$ is infinite and nonrepeating or else the sequence is repeating with period n where n is not prime. We proceed the same way in both cases: let $x * y = xy^q$ where q is a prime number chosen according to the following rules.

- $q = 2$ if Σ is nonrepeating.
- q is the least prime divisor of n if Σ is repeating of period n.

Because q is prime, our first assumption (that the claim is false) implies that we do not have $u = u * v = uv^q$. We cannot have $v = u * v$ either, since then

$$uv^q = u * v = v = vv^q$$

contradicts either the right cancellativity of xy or else the fact that $u \neq v$. Thus, $x * y$ is nontrivial and must generate the full clone of $\mathbf{A}$. In particular, since $uv \in \mathrm{Sg}^{\mathbf{A}}(\{\mathrm{u}, \mathrm{v}\})$, there must be a term $b(x, y)$ in the clone generated by $x * y$ such that $b(u, v) = uv$. However, we shall see that there is no such term.

The operation $x * y$ also satisfies $x * (x * y) = x$, as well as all other properties ascribed to xy. Hence terms generated by $x * y$ have the same form as those generated by xy: a binary term may be written as a left–associated product of the form $x * y^{*k}$ for some k. Thus, if there is a $*$–term $b(x, y)$ such that $b(u, v) = uv$, then $b(x, y) = x * y^{*k}$ or else $b(x, y) = y * x^{*k}$. Translating back via the definition $x * y = xy^q$, we get that $b(x, y) = xy^{qk}$ or yx^{qk} for some k. However,

$$uv = b(u, v) = uv^{qk}$$

is impossible when Σ is nonrepeating. If Σ is repeating, then the displayed equation implies that the period of Σ divides $qk - 1$. But q divides the period of Σ, so q must divide $qk - 1$ which is an impossibility. The case

$$uv = b(u, v) = vu^{qk}$$

is likewise impossible because it leads to

$$u = u(uv) = u(vu^{qk}) = uv$$

and we began with the assumption that $u \neq uv$. Thus, we get a contradiction unless Σ has prime period. If that prime is p, then $uv^p = u$. This finishes the proof of the claim.

Note that the term condition applied to

$$\underline{u}v^k = u = \underline{u}u^k \Leftrightarrow \underline{v}v^k = v = \underline{v}u^k$$

yields the conclusion that the length of the period of the sequence $\Sigma = (u, uv, uv^2, \dots)$ is the same as the length of the period of $\Sigma' = (v, vu, vu^2, \dots)$. Thus $vu^p = v$ for the same prime p as in the claim. Using this it is not hard to completely describe the operation table for $\langle A; xy\rangle$ and, in particular, to see that $\mathbf{A} \models xy^p = x$.

The operation xy on $\mathbf{A}$ satisfies the equations

$$xx = x,\ x(yz) = xy,\ (xy)z = (xz)y, \text{ and } xy^p = x.$$

The variety axiomatized by a binary operation satisfying these equations has a minimal clone since, modulo these equations, any term may be written as a left–associated expression $xy_1^{k_1} \cdots y_r^{k_r}$ with $0 \leq k_i < p$. If such a term is not projection onto x, then some $k_i > 0$ and we can specialize to $x * y_i := xy_i^{k_i}$ by setting $y_j = x$ for all $j \neq i$. But then $xy = x * y^{*\ell}$ for any ℓ such that $k_i \cdot \ell \equiv 1(\text{mod } p)$. Thus, any term different from a projection generates xy.

Any member of the variety axiomatized by the previously listed equations which is not a left zero semigroup generates the entire variety. To show this, use the normal form for terms to show that any additional equation implies $x = xy$. In particular, $\mathbf{A}$ generates this variety. We get that every generator of the variety is a nontrivial representation of $\mathcal{C}$, since $\mathbf{A}$ is. The algebra $\mathbf{P}$ described in the statement of the lemma satisfies the above equations, is abelian and is not a left zero semigroup. It follows that $\mathbf{P}$ is a nontrivial abelian representation of $\mathcal{C}$. □

In [5], one finds a description of the idempotent subclones of a cyclic group of prime power order. Instead of giving a direct argument, as we did, we could have established the minimality of the clone of $\mathbf{P}$ by just consulting this description.

Now we know at least one nontrivial abelian representation for each minimal clone of class (II) that has such a representation. We use this information to describe all minimal clones that have a nontrivial abelian representation. So far, we know that such clones are unary, of class (II), or isomorphic to the clone of an affine vector space on a two–element set. Our main tool is the following lemma which we have abstracted from the proof of Lemma 3.4 of [8].

LEMMA 3.6. *Let* $\mathbf{A}$ *be a nontrivial representation of a minimal clone* $\mathcal{C}$ *and assume that* $t(\bar{x})$ *is a term in the language of* $\mathbf{A}$*. If* $\mathbf{A}$ *satisfies the equation* $t(\bar{x}) = x_i$ *for some* i*, then every representation of* $\mathcal{C}$ *satisfies this equation.*

Proof. Assume otherwise that $\mathbf{B}$ is a representation which does not satisfy the equation $t(\bar{x}) = x_i$. Then $\mathbf{A} \times \mathbf{B}$ is a representation of $\mathcal{C}$ where the operation $t^{\mathbf{A}\times\mathbf{B}}(\bar{x})$ is not a projection. Hence the clone of $\mathbf{A} \times \mathbf{B}$ is generated by $t^{\mathbf{A}\times\mathbf{B}}(\bar{x})$. By factoring we get that the clone of $\mathbf{A}$ is generated by $t^{\mathbf{A}}(\bar{x}) = x_i$. But this is impossible since the clone of $\mathbf{A}$ is nontrivial. $\square$

The previous lemma shows that if we know one nontrivial representation of a minimal clone, then we can deduce a lot about the equations that hold in other representations. An equation of the form $t(\bar{x}) = x_i$, which asserts that an operation $t(\bar{x})$ interprets as projection onto a variable, will be called an *absorption equation*.

COROLLARY 3.7. *Let* $\mathbf{A}$ *be a nontrivial representation of a minimal clone* $\mathcal{C}$*. If* $\mathbf{A}$ *is axiomatizable by absorption equations, then* $\mathbf{A}$ *is a faithful representation of* $\mathcal{C}$*.*

Proof. If $\mathcal{V}$ is the variety of representations of $\mathcal{C}$, then the hypotheses imply that $\mathbf{A} \in \mathcal{V}$ and (from Lemma 3.6) $\mathcal{V} \subseteq \mathsf{HSP}(\mathbf{A})$. Therefore $\mathcal{V} = \mathsf{HSP}(\mathbf{A})$, which implies that the representation $\mathbf{A}$ is faithful. $\square$

Suppose that f and g both generate the minimal clone $\mathcal{C}$. If some f–representation $\mathbf{A} = \langle A; f\rangle$ is axiomatizable by absorption equations, then is it true that the corresponding g–representation $\langle A; g\rangle$ is also axiomatizable by absorption equations? We don't know. Since f and g both generate $\mathcal{C}$, there is an expression for either of f or g in terms of the other: say $f = \varphi(g)$ and $g = \phi(f)$ where φ and ϕ are terms in the language of clones. Now suppose that F is a set of absorption equations, written only in terms of f, which axiomatizes $\langle A; f\rangle$. One can use $f = \varphi(g)$ to transform F into a set G of absorption equations written only in terms of g. The question becomes, does G axiomatize $\langle A; g\rangle$?

The statement that F axiomatizes $\langle A; f\rangle$ implies that

$$F \models f(\bar{x}) = \varphi\phi(f)(\bar{x}).$$

The set $G \cup \{g(\bar{x}) = \phi\varphi(g)(\bar{x})\}$ axiomatizes $\langle A; g\rangle$. If we could show that

$$(F \models f(\bar{x}) = \varphi\phi(f)(\bar{x})) \Rightarrow (G \models g(\bar{x}) = \phi\varphi(g)(\bar{x})),$$

then $\langle A; g\rangle$ would be axiomatizable by the absorption equations in G. We don't know if this implication holds. (Nevertheless, we know of no example where a minimal clone generated by either f or g has a nontrivial f–representation axiomatizable by absorption equations which is not so axiomatizable as a g–representation.) The impact of this observation is that we shall specify the generating operation for the clone when we claim that a representation has an axiomatization by absorption equations.

LEMMA 3.8. *Assume that a minimal clone has a rectangular band as a nontrivial abelian representation. The representation is faithful.*

Proof. Assume that $\mathcal{C}$ is a clone satisfying the hypotheses and that $\mathbf{A}$ is a nontrivial representation of $\mathcal{C}$ which is a rectangular band. We consider $\mathcal{C}$ to be a clone of class (II) such that $\mathbf{A} = \langle A; xy\rangle$ is a rectangular band with a nontrivial clone. The defining equations for rectangular bands are:

$$\begin{aligned} x &= xx, \\ (xy)z &= x(yz), \\ xz &= xyz. \end{aligned}$$

This is not an axiomatization by absorption equations for the variety of rectangular bands, but such an axiomatization exists. Here is one:

$$\begin{aligned} x &= xx, \\ x &= x(y(zx)), \\ x &= ((xy)z)x. \end{aligned}$$

To prove that these absorption equations axiomatize the variety of rectangular bands, let $\mathcal{U}$ be the variety of groupoids axiomatized by these equations. All rectangular bands satisfy these equations, so $\mathcal{U}$ contains the variety of rectangular bands. We shall argue that $\mathcal{U} \models xz = x(yz)$. By symmetry, we will get $\mathcal{U} \models xz = (xy)z$ and therefore that

$$\mathcal{U} \models x(yz) = xz = (xy)z.$$

This will prove that $\mathcal{U}$ satisfies all the usual defining axioms for the variety of rectangular bands.

Claim. $\mathcal{U} \models (XY = XZ) \;\&\; (YU = ZU) \;\Rightarrow Y = Z$.

Proof of Claim. Choose $\mathbf{B} \in \mathcal{U}$ and $a, b, c, d \in B$ such that $ac = ad$ and $cb = db$. $\mathcal{U}$ is assumed to satisfy the equations $x = x(y(xx)) = x(yx)$ and $x = x(y(zx))$. They can be applied as follows:

$$cd = c[\underbrace{d(ad)}] = \underbrace{c[d(ac)]} = c.$$

Now working in the other direction we can use the equations $x = ((xx)z)x = (xz)x$ and $x = ((xy)z)x$ to obtain that

$$cd = (\underbrace{(cb)c})d = \underbrace{((db)c)d} = d.$$

Hence $c = d$ and the claim is proven.

Now we show that the equation $x(yz) = xz$ holds throughout $\mathcal{U}$. For $X := z$, $Y := x(yz)$, $Z := xz$ and $U := x$ we have $XY = z(x(yz)) = z = z(x(zz)) = z(xz) = XZ$. The equation $((xy)z)x = x$ has the consequence $(xz)x = ((xx)z)x = x$. This gives the second and third equalities in $YU = (x(yz))x = x = (xz)x = ZU$. By the previous claim we must have

$$x(yz) = Y = Z = xz.$$

This holds for arbitrary x, y and z. A similar argument shows that the equation $(xy)z = xz$ holds throughout $\mathcal{U}$. Hence $x(yz) = xz = (xy)z$ and $x = xx$ are equations of $\mathcal{U}$, while at the same time $\mathcal{U}$ contains all groupoids satisfying these equations. It follows that the absorption equations listed in this proof are an axiomatization of the variety of rectangular bands. Now this lemma is a consequence of Corollary 3.7. □

LEMMA 3.9. *Assume that a minimal clone has an affine vector space over a prime field as a nontrivial abelian representation. The representation is faithful.*

Proof. Let $\mathbf{A}$ be a nontrivial representation of $\mathcal{C}$ as an affine vector space over a field with q elements where q is the appropriate prime. We choose our generator for $\mathcal{C}$ to be any ternary operation $p(x, y, z)$ which interprets as a Mal'cev operation in $\mathbf{A}$. The Mal'cev operation of an affine vector space is unique: it is $p^{\mathbf{A}}(x, y, z) = x - y + z$. Clearly, with respect to this operation, $\mathbf{A}$ satisfies the following absorption equations:

$$\begin{aligned}
x &= p(x, y, y),\\
x &= p(y, y, x),\\
x &= p(p(z, y, x), z, y),\\
x &= p(p(p(x, y, z), z, u), u, y),\\
x &= \underbrace{p(p(\cdots(p}_{q \text{ times}}(x, y, z), y, z), \cdots), y, z).
\end{aligned}$$

However, these equations axiomatize $\mathbf{A}$. (A proof can be found on page 266 of [8].) Corollary 3.7 can now be invoked to prove that $\mathbf{A}$ is a faithful representation of $\mathcal{C}$. □

LEMMA 3.10. *Assume that, for some prime p, a minimal clone has the algebra $\mathbf{P}$ from Lemma 3.5 as a nontrivial abelian representation. The representation is faithful.*

Proof. The variety generated by $\mathbf{P}$ is axiomatizable by the following list of absorption equations:

$$\begin{aligned} x &= xx, \\ x &= xy^p, \\ x &= (xy^{p-1})(yz), \\ x &= (((xz)y)z^{p-1})y^{p-1}. \end{aligned}$$

(Where we omit necessary parentheses, as in xy^p, we mean a left–associated product.)

It is easy to check that the listed absorption equations hold in $\mathbf{P}$. To show that they axiomatize $\mathbf{P}$, recall from Lemma 3.5 that $\mathbf{P}$ is axiomatized by

$$xx = x,\ x(yz) = xy, (xy)z = (xz)y, \text{ and } xy^p = x.$$

We only need to show how to deduce $x(yz) = xy$ and $(xy)z = (xz)y$ from our list of absorption equations. For the first of these, let $u = xy$. Then $uy^{p-1} = (xy)y^{p-1} = xy^p = x$. Therefore we have

$$x(yz) = (uy^{p-1})(yz) = u = xy.$$

For the second equation, multiply both sides of

$$x = (((xz)y)z^{p-1})y^{p-1}$$

on the right by y first and then by z. The result obtained after simplifying is that $(xy)z = (xz)y$. This finishes the proof. □

THEOREM 3.11. *The minimal clones which have a nontrivial abelian representation are the following:*

(i) *the unary clone generated by an operation f satisfying $f(x) = f(y)$, but not satisfying $f(x) = x$;*
(ii) *the unary clone generated by an operation f satisfying $f^2(x) = f(x)$, but not satisfying $f(x) = f(y)$ or $f(x) = x$;*
(iii) *the unary clone generated by an operation f satisfying $f^p(x) = x$ for some prime p, but not satisfying $f(x) = x$;*
(iv) *the clone of any rectangular band not equal to a left or right zero semigroup;*
(v) *the clone of an affine vector space over a prime field;*
(vi) *the clone of the algebra $\mathbf{P}$ for some prime p (defined in Lemma 3.5).*

□

COROLLARY 3.12. *A minimal clone has a nontrivial abelian representation iff it has a faithful representation as a reduct of a finite cyclic group.*

Proof. Referring to the cases enumerated in Theorem 3.11:

(i) Take the reduct of $\mathbf{Z}_2$ to $f(x) = 0$.
(ii) Take the reduct of $\mathbf{Z}_6$ to $f(x) = 3x$.
(iii) Take the reduct of $\mathbf{Z}_{2^p-1}$ to $f(x) = 2x$.
(iv) Take the reduct of $\mathbf{Z}_6$ to $xy = 3x + 4y$.
(v) Take the reduct of $\mathbf{Z}_p$ to $p(x, y, z) = x - y + z$.
(vi) Take the reduct of $\mathbf{Z}_{p^2}$ to $xy = (1 - p)x + py$.

We remark that in case (v) the choice $p = 2$ yields the unique minimal clone in class (IV), while for $p > 2$ we get clones in class (II) with $xy = p(x, y, x) = 2x - y$ as (one choice for) the generator of the clone. □

As an application of Theorem 3.11, we answer Pálfy's question: Which Mal'cev operations generate minimal clones?

THEOREM 3.13. *Let $\mathcal{C}$ be a minimal clone generated by a Mal'cev operation. Then $\mathcal{C}$ has a nontrivial abelian representation. Hence, $\mathcal{C}$ is the clone of an affine vector space over a prime field.*

Proof. Let $\mathcal{V}$ be the variety of p–representations, where $p = p(x, y, z)$ is the Mal'cev operation of $\mathcal{C}$. $\mathcal{V}$ is an idempotent, congruence permutable (CP) variety. If $\mathcal{V}$ were congruence distributive (CD), then (since $\mathcal{V}$ is also CP) we would have that $\mathcal{V}$ is arithmetical. It is known that any arithmetical variety has a Pixley term. But $\mathcal{V}$ has a minimal clone and no minimal clone contains a Pixley term as we observed after Theorem 2.2. This shows that $\mathcal{V}$ is not CD.

From basic commutator theory (see [2]), a CP variety which is not CD contains a member with a nonzero abelian congruence. Since $\mathcal{V}$ is idempotent, any congruence class of an algebra is a subuniverse. Because of the way the commutator restricts to subalgebras, any class of an abelian congruence on some member of $\mathcal{V}$ generates an abelian subalgebra. Hence, the fact that $\mathcal{V}$ is not CD implies that $\mathcal{V}$ contains a nontrivial abelian algebra, $\mathbf{A}$. The clone $\mathcal{C}$ has an abelian representation $\mathbf{A}$ which is nontrivial (since $\mathbf{A}$ has a Mal'cev operation in its clone). This proves the first claim of the theorem.

The second claim follows from the fact that $\mathcal{C}$ has a Mal'cev operation and is on the list of clones from Theorem 3.11. This forces $\mathcal{C}$ to be one of the clones described in Theorem 3.11 (v). □

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