CORRIGENDUM

CLONES OF FINITE GROUPS

KEITH A. KEARNES AND ÁGNES SZENDREI

1. The Clone of the Quaternion Group

We proved in *Clones of finite groups*, [1], that any operation on the 8-element quaternion group Q_8 that preserves all the 5-ary algebraic relations of Q_8 is in the clone of Q_8, but there is an operation that preserves all the 3-ary algebraic relations that is not in the clone. Nothing is proved in [1] about the operations on Q_8 that preserve the 4-ary algebraic relations, but we claimed at the bottom of page 50 of [1] that there is an operation that preserves all the 4-ary algebraic relations that is not in the clone. We later realized that was false and published the correction in [2].

2. Commutator Collection

One of the proofs in [1] could benefit from further explanation:

Claim 3.15. If G is a nilpotent group of class c, $t(x_1, \ldots, x_n)$ is an n-ary term operation of G for some $n > c$, and $t[x_i/1]$ is constant for all i, then t is constant. (Here $t[x_i/1]$ represents the term obtained from t by replacing x_i with 1.)

We stated that this claim follows from commutator collection. Here we explain in more detail what that means.

Recursively define the set C of *commutator words* in a given set of variables to be the smallest set of group words containing 1, the variables and their inverses, and closed under the formation of commutators (which means that $w_1, w_2 \in C \to [w_1, w_2] \in C$). Here $[x, y] = x^{-1}y^{-1}xy$, so

$$w_2 w_1 = w_1 w_2 [w_2, w_1] \tag{2.1}$$

holds for any words w_1 and w_2. Define the *weight* of a commutator word w to be the number of occurrences of variables or their inverses when w is written as a commutator word, so for example $[[x^{-1}, [x, y]], [z, x]]$ is a commutator word of weight 5.

Lemma 1. *Let $X = \{x_1, \ldots, x_n\}$ be a set of variables, and let $<$ be a linear order of the subsets of X which extends the subset order. If $t(x_1, \ldots, x_n)$ is a group word in the variables X, then t may be factored as*

$$t(x_1, \ldots, x_n) = T_{U_0} T_{U_1} \cdots T_{U_k} \tag{2.2}$$

where $U_0 < U_1 < \cdots < U_k$ is the previously fixed linear order of the subsets of X and each T_{U_i} is a product of commutator words whose variables are precisely those in U_i.

Proof. Since the linear order $U_0 < U_1 < \cdots < U_k$ extends the subset order, we may assume after relabeling variables that it begins $\emptyset < \{x_1\} < \{x_2\} < U_3 < \cdots$. Write t as a product of variables and their inverses. Move the leftmost occurrence of x_1 all the way to the left using equation (2.1), for example

$$\begin{aligned} x_3(x_7^{-1}x_1)x_1x_2\ldots &= x_3(x_1x_7^{-1}[x_7^{-1},x_1])x_1x_2\ldots \\ &= (x_3x_1)x_7^{-1}[x_7^{-1},x_1]x_1x_2\ldots \\ &= (x_1x_3[x_3,x_1])x_7^{-1}[x_7^{-1},x_1]x_1x_2\ldots \\ &= x_1x_3[x_3,x_1]x_7^{-1}[x_7^{-1},x_1]x_1x_2\ldots. \end{aligned}$$

Then move the next leftmost occurrence of x_1, and repeat until all x_1's are on the left. The word now begins with $x_1^{e_1}$, which will be $T_{\{x_1\}}$. We call this the *collected part.* The rest of the word, to the right, is the *uncollected part.* Apply the same algorithm to x_2 to obtain a collected part of the form $x_1^{e_1}x_2^{e_2} = T_{\{x_1\}}T_{\{x_2\}}$. After we have a collected part of the form $T_{U_0}\cdots T_{U_i}$, apply the same procedure to the commutator words in the uncollected part whose variables are precisely those in U_{i+1} working from leftmost occurrence to rightmost occurrence. This process introduces new commutator words to the uncollected part, but the set of variables in an introduced word properly contains U_{i+1} so these words will be handled at a later stage. □

In [1], we derive Claim 3.15 from "commutator collection". It is the form of collection in the above lemma that is needed. Namely, if $t[x_i/1] = 1$ for all i, then one can show by induction that $T_{U_j} = 1$ for all $j < k$. (After one has shown $T_{U_0} = \cdots = T_{U_i} = 1$, substitute 1 for all x_i not in U_{i+1} to obtain $T_{U_{i+1}} = 1$.) This proves that if $t[x_i/1] = 1$ for all i, then $t = T_{U_k}$ is a product of commutator words where every variable in X appears in every commutator word in the product. In particular, every commutator word in the product has weight at least n. Now when G is a nilpotent group of class $c < n$, we get that $t = T_{U_k} = 1$, since all commutator words whose weight exceeds the nilpotence class are 1.

References

[1] K. A. Kearnes and Á. Szendrei, *Clones of finite groups*, Algebra Universalis **54** (2005), no. 1, 23–52.

[2] K. A. Kearnes, Jason Shaw and Á. Szendrei, *Clones of 2-step nilpotent groups*, Algebra Universalis (to appear).

[3] Jason Shaw, *Commutator Relations and the Clones of Finite Groups*, Ph.D. dissertation, University of Colorado at Boulder, 2008.

(Keith Kearnes) Department of Mathematics, University of Colorado, Boulder, CO 80309-0395, U.S.A.

E-mail address: Keith.Kearnes@Colorado.EDU

(Ágnes Szendrei) Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H–6720 Szeged, Hungary, and Department of Mathematics, University of Colorado, Boulder, CO 80309-0395, U.S.A.

E-mail address: a.szendrei@math.u-szeged.hu, szendrei@euclid.colorado.edu