Congruence Lattices of Congruence Semidistributive Algebras

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Dedicated to Garrett Birkhoff on the occasion of his eightieth birthday

Nearly twenty years ago, two of the authors wrote a paper on congruence lattices of semilattices [9]. The problem of finding a really useful characterization of congruence lattices of finite semilattices seemed too hard for us, so we went on to other things. Thus when Steve Seif asked one of us at the October 1990 meeting of the AMS in Amherst what we had learned in the meantime, the answer was nothing. But Seif's question prompted us to return to the subject, and we soon found that we had missed at least one nice property: *the congruence lattice of a finite semilattice is an upper bounded homomorphic image of a free lattice.* This strengthens the well-known fact that congruence lattices of semilattices satisfy the meet semidistributive law $\mathrm{SD}_\wedge$.

It turns out that this result admits a striking generalization: *if* $\mathbf{V}$ *is a variety of algebras whose congruence lattices are meet semidistributive, then the congruence lattices of finite algebras in* $\mathbf{V}$ *are upper bounded homomorphic images of a free lattice.* The proof of this theorem takes us into the realm of tame congruence theory, and with modest additional effort we are able to find strong restrictions on the structure of the lattice $\mathcal{L}_v(\mathbf{W})$ of subvarieties of an arbitrary locally finite variety $\mathbf{W}$.

Stimulated by Viktor Gorbunov's talk at the Jónsson Symposium in Iceland in July 1990, and the corresponding draft of [12] which he provided us, we went on to ask if this type of result might apply to lattices of quasivarieties. It is known that the lattice $\mathcal{L}_q(\mathbf{K})$ of all quasivarieties contained in a quasivariety $\mathbf{K}$ satisfies $\mathrm{SD}_\vee$ [11], and the improved (finite) version states that *if* $\mathbf{K}$ *is a locally finite quasivariety of finite type and* $\mathcal{L}_q(\mathbf{K})$ *is finite, then it is a lower bounded homomorphic image of a free lattice.* There are natural generalizations of this theorem for varieties which are not locally finite.

Perhaps an analogy with the modular and Arguesian laws provides a good way to interpret these results. Dedekind devised the modular law to capture the permutability of normal subgroup lattices, but the Arguesian law is now recognized to be a more accurate reflection of this property. Similarly, congruence lattices of

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semilattices satisfy $\mathrm{SD}_\wedge$, but (at least in the finite case) upper boundedness is a stronger property which provides a better description of their structure. As the Arguesian law is not sufficient to characterize normal subgroup lattices, neither does lower boundedness characterize congruence lattices of finite semilattices. Nonetheless, the Arguesian law and upper boundedness, respectively, play a significant role in refining our understanding of these classes of lattices.

Since completing the draft of this paper, we learned of significant progress on the Siberian front. In particular, K. V. Adaricheva has found a characterization of subalgebra lattices of finite semilattices [1], and her results imply some of our results. Likewise, Gorbunov was aware of our elementary theorems on quasivariety lattices, as consequences of his deeper work on the subject. The main new results in this paper are in Theorems 23, 31 and 42. For purposes of exposition, we have decided to keep the proofs of some of the overlapping results in this paper, but we will try to indicate when these results were found independently.

1. Lower bounded lattices

Let us begin by reviewing the basic definition and properties of finite lower bounded lattices. These come from [5], [16], and [21]; other fundamental results about this class can be found in [10] and [24]. The dual notion is called upper bounded.

We need a series of definitions.

A lattice homomorphism $h : \mathcal{K} \to \mathcal{L}$ is called a *lower bounded homomorphism* if for each $a \in L$, $\{x \in K : h(x) \geq a\}$ is either empty or else has a least element, denoted $\beta(a)$. Note that the domain of β is an ideal $\mathcal{I}$ of $\mathcal{L}$, and $\beta : \mathcal{I} \to \mathcal{K}$ is a join homomorphism.

A subset $\mathrm{C} \subseteq K$ is a *lower pseudo-interval* if it is a union of intervals with a common least element, $\mathrm{C} = \bigcup [a, b_i]$. If $\mathrm{C} \subseteq K$ is a lower pseudo-interval, the Day doubling construction yields the lattice $\mathcal{K}[\mathrm{C}]$ with universe $K - \mathrm{C} \cup (\mathrm{C} \times \{0, 1\})$, endowed with the natural order. Let $\mathbf{LD}$ denote the smallest class of finite lattices containing all finite distributive lattices and closed under the doubling of lower pseudo-intervals.

If $\mathcal{L}$ is a finite lattice, let $J(\mathcal{L})$ denote the set of (nonzero) join irreducible elements of $\mathcal{L}$. For subsets $A, B \subseteq \mathcal{L}$, define $A \ll B$ if for each $a \in A$ there exists $b \in B$ with $a \leq b$. Using this, for $k \in \omega$ we define subsets $D_k(\mathcal{L}) \subseteq J(\mathcal{L})$ as follows. $D_0(\mathcal{L})$ is the set of all join-prime elements of $\mathcal{L}$. Given $D_k(\mathcal{L})$, we define $D_{k+1}(\mathcal{L})$ to be the set of all $p \in J(\mathcal{L})$ such that whenever $p \leq \bigvee B$ and $p \nleq b$ for all $b \in B$, then there exists $A \ll B$ such that $p \leq \bigvee A$ and $A \subseteq D_k(\mathcal{L})$.

Closely related to the subsets $D_k(\mathcal{L})$ is the *dependence relation* D on $J(\mathcal{L})$. For distinct elements $p, q \in J(\mathcal{L})$, let $p \, D \, q$ if there exists $x \in \mathcal{L}$ such that $p \leq q \vee x$ but $p \nleq q_* \vee x$, where q_* denotes the unique lower cover of q. A *D-cycle* in $\mathcal{L}$ is a sequence $p_0, p_1, \ldots, p_{n-1}$ of distinct elements in $J(\mathcal{L})$ such that

$$p_0 \, D \, p_1 \, D \, \ldots \, D p_{n-1} \, D \, p_0 \, .$$

The dual of the dependence relation, defined on $M(\mathcal{L})$, will be denoted by D^d.

With these definitions assembled, we can state the basic theorem.

Theorem 1. *For a finite lattice $\mathcal{L}$, the following are equivalent.*

(1) *There exists a finite set X and a lower bounded epimorphism $f : FL(X) \twoheadrightarrow \mathcal{L}$.*
(2) *For every finitely generated lattice $\mathcal{K}$, every homomorphism $h : \mathcal{K} \to \mathcal{L}$ is lower bounded.*
(3) *$\mathcal{L} \in$ **LD**, i.e., $\mathcal{L}$ can be obtained from a distributive lattice by a sequence of doublings of lower pseudo-intervals.*
(4) *$D_k(\mathcal{L}) = J(\mathcal{L})$ for some $k \in \omega$.*
(5) *$\mathcal{L}$ contains no D-cycle.*

A finite lattice is called *lower bounded* if it satisfies these properties, and *upper bounded* if it has the dual properties. A lattice which is both upper and lower bounded is called *bounded.* If $\mathcal{L}$ is lower bounded and $p \in J(\mathcal{L})$, the *D-rank* of p is the least r such that $p \in D_r(\mathcal{L})$.

It is not hard to see, using (1) and the fact that β is a join homomorphism, that a lower bounded lattice inherits the property

$$(\mathrm{SD}_\vee) \qquad x \vee y = x \vee z \text{ implies } x \vee y = x \vee (y \wedge z)$$

from free lattices. Likewise, an upper bounded lattice satisfies the dual property $(\mathrm{SD}_\wedge)$.

The lattice of convex subsets of a four-element chain (Figure 1) provides an example of a lattice which satisfies $(\mathrm{SD}_\vee)$ but is not lower bounded.

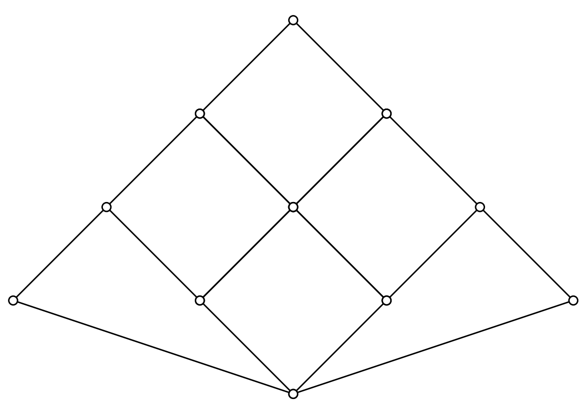

FIGURE 1: **Co 4**

At one point we get to use a beautiful result of Alan Day [5] (see [16], [23]).

Theorem 2. *A finite lower bounded lattice which satisfies* $\mathrm{SD}_\wedge$ *is also upper bounded.*

2. Congruence lattices of finite semilattices

The next lemma, from our old paper [9], provides a useful tool for working with congruence lattices of semilattices. Note that if $\mathcal{T} = \langle T; \wedge\rangle$ is a finite meet semilattice and $a, b \in T$ are elements with a common upper bound, then they have a least upper bound, which we will denote by $a + b$. Thus $+$ is in general a partial operation on T, and $\mathcal{T}^* = \langle T; +, 0\rangle$ is a partial algebra with a constant.

Lemma 3. *If* T *is a finite meet semilattice, then* $\mathbf{Con}\ \mathcal{T}$ *is dually isomorphic to the subalgebra lattice* $\mathbf{Sub}\ \mathcal{T}^*$.

Indeed, the $+$–subalgebra corresponding to a congruence θ is the set of all minimum elements of θ-classes.

It is convenient to always work using this duality. In these terms, the old result is that $\mathbf{Sub}\ \mathcal{T}^*$ is a point lattice satisfying $\mathrm{SD}_\vee$; this implies that it is dually semimodular. The basic new claim (also in Adaricheva [1]) is the following.

Theorem 4. $\mathbf{Sub}\ \mathcal{T}^*$ *is a lower bounded lattice.*

We will give two proofs, the first using property (5) of Theorem 1, and the second using (3).

Proof 1. It is easy to see that the join irreducible subalgebras of $\mathcal{T}^*$ are exactly those of the form $\overline{a} = \{a, 0\}$ for $0 \neq a \in T$. Moreover, in $\mathbf{Sub}\ \mathcal{T}^*$ we have $\overline{a} \leq X \vee Y$ if and only if $a = x + y$ for some $x \in X$, $y \in Y$. Hence $\overline{a}\, D\, \overline{b}$ if and only if $a = b + x$ for some $x \in T$ with b and x incomparable. In particular, $\overline{a}\, D\, \overline{b}$ implies $a > b$, so there are no D-cycles in $\mathbf{Sub}\ \mathcal{T}^*$. By (5), $\mathbf{Sub}\ \mathcal{T}^*$ is lower bounded. $\square$

Refining this argument, one can show by induction that if a is an element of height k in $\mathcal{T}$, then $\overline{a} \in D_{k-1}(\mathbf{Sub}\ \mathcal{T}^*)$, so that property (4) holds.

Proof 2. Let t be maximal in T. Then we can map $\mathbf{Sub}\ \mathcal{T}^*$ onto $\mathbf{Sub}\ (\mathcal{T} - \{t\})^*$ by $\rho(A) = A \cap (T - \{t\})$. It is straightforward to check, using the maximality of t, that ρ is a homomorphism. A subalgebra B of $\mathcal{T} - \{t\}$ has one or two preimages, depending on whether or not t is a join of elements of B. Thus $\mathbf{Sub}\ \mathcal{T}^*$ is obtained from $\mathbf{Sub}\ (\mathcal{T} - \{t\})^*$ by doubling all the subalgebras B which are also subalgebras of $(\mathcal{T} - \{t\})^*$, and this is a union of ideals. (Note that every atom gets doubled!) By induction $\mathbf{Sub}\ (\mathcal{T} - \{t\})^* \in \mathbf{LD}$, and hence $\mathbf{Sub}\ \mathcal{T}^* \in \mathbf{LD}$. $\square$

Combining Lemma 3 and Theorem 4, we obtain the desired result for congruence lattices of semilattices.

Corollary 5. *If $\mathcal{S}$ is a finite semilattice, then* $\mathbf{Con}\,\mathcal{S}$ *is an upper bounded lattice.*

K. V. Adaricheva proved Theorem 4 independently [1], and added a partial converse: *every finite lower bounded lattice can be embedded into* $\mathbf{Sub}\,\mathcal{S}$ *for some finite semilattice* $\mathcal{S}$. Note that her theorem implies our result in [9] that congruence lattices of semilattices satisfy no nontrivial lattice identity. Following this line of thought, V. B. Repnitzkii has announced the following extended version of this result.

Theorem 6. *The following are equivalent for a finite lattice* $\mathcal{L}$.

(1) $\mathcal{L}$ *is lower bounded.*
(2) $\mathcal{L}$ *is embeddable in the subsemigroup lattice of a free semigroup.*
(3) $\mathcal{L}$ *is embeddable in the subsemigroup lattice of a free commutative semigroup.*
(4) $\mathcal{L}$ *is embeddable in the subsemigroup lattice of an infinite cyclic group.*
(5) $\mathcal{L}$ *is embeddable in the subsemigroup lattice of a finite semilattice.*
(6) $\mathcal{L}$ *is embeddable in the subsemigroup lattice of a finite nilpotent semigroup.*

As mentioned earlier, Adaricheva also characterized those lattices which can be represented as $\mathbf{Sub}\ \mathcal{S}$ with $\mathcal{S}$ a finite semilattice ([1], see also [2]). However, there are partial join semilattices $\mathcal{T} = \langle T; +, 0\rangle$ such that $\mathbf{Sub}\ \mathcal{T}$ cannot be represented as $\mathbf{Sub}\ \mathcal{S}$ for any semilattice $\mathcal{S}$, the simplest such example being the partial join semilattice given in Figure 2. So her characterization of the subalgebra lattices of finite semilattices does not automatically give a dual characterization of their congruence lattices. As far as we know, this problem remains open, though it is quite likely that the same methods will apply.

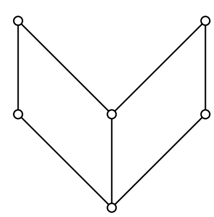

FIGURE 2

Seif's original question was prompted by his investigation of congruence lattices of certain types of semigroups. In particular, he has found a nice description of $\mathbf{Con}\,\mathcal{S} \times \mathcal{G}$, where $\mathcal{S}$ is a finite semilattice and $\mathcal{G}$ a finite group, and the product is regarded as a semigroup [28]. Using this, we can extend Corollary 5 as follows.

Theorem 7. *If $\mathcal{S}$ is a finite semilattice and $\mathcal{G}$ a finite group with* $\mathbf{Con}\,\mathcal{G}$ *distributive, then* $\mathbf{Con}\,\mathcal{S} \times \mathcal{G}$ *is an upper bounded lattice.*

Perhaps more naturally, Corollary 5 can also be applied to the trace and kernel decomposition of congruences of an inverse semigroup.

Corollary 8. *If $\mathcal{S}$ is a finite inverse semigroup, then the lattice of normal congruences of $E_{\mathcal{S}}$ is an upper bounded lattice.*

Corollary 9. *Let $\mathbf{B}_n$ be the variety of all inverse semigroups satisfying $x^{n+1} \approx x^n$. For each finite semigroup $\mathcal{S} \in \mathbf{B}_n$, $\mathbf{Con}\,\mathcal{S}$ is an upper bounded lattice.*

3. A quick review of tame congruence theory

In order to generalize Corollary 5, we will need to make heavy use of tame congruence theory. The primary reference for this is of course the book of David Hobby and Ralph McKenzie [14]. In this section we will briefly review the parts of tame congruence theory which we will need in the sequel; except where otherwise noted, everything in this section comes directly from [14].

Throughout this section, all algebras are finite. Recall that given an algebra $\mathcal{A}$ and a subset $U \subseteq \mathcal{A}$, we can form the algebra $\mathcal{A}|_U$ with universe U, whose operations are all $f \in \mathrm{Pol}\ \mathcal{A}$ with respect to which U is closed. Let $E = \{e \in \mathrm{Pol}_1\ \mathcal{A} : e^2 = e\}$. The following famous fundamental fact is from P. P. Pálfy and P. Pudlák [25] and, for later reference, does *not* require that $\mathcal{A}$ be finite.

Theorem 10. *Let $U = e(A)$ for some $e \in E$. Then the natural restriction map*

$$\rho : \mathbf{Con}\,\mathcal{A} \to \mathbf{Con}\,\mathcal{A}|_U$$

given by $\rho(\theta) = \theta|_U$ is a complete lattice epimorphism.

For $\alpha < \beta$ in $\mathbf{Con}\,\mathcal{A}$, let

$$U(\alpha, \beta) = \{f(A) : f \in \mathrm{Pol}_1\ \mathcal{A} \text{ and } f(\beta) \nsubseteq \alpha\}$$

and let $M(\alpha, \beta)$ denote the collection of minimal members of $U(\alpha, \beta)$ with respect to set inclusion. These latter are called *$\langle \alpha, \beta\rangle$-minimal sets.* Clearly, if $U \in M(\alpha, \beta)$ and f is a unary polynomial with $f(U) \subseteq U$, then either $f|_U$ is a permutation or $f(\beta|_U) \subseteq \alpha|_U$.

For our purposes, it is not necessary to recall the definition of a tame quotient, but only to know that they are a generalization of prime quotients.

Lemma 11. *Prime quotients are tame in* $\mathbf{Con}\,\mathcal{A}$.

Let B, C be nonempty subsets of A. We say that B and C are polynomially isomorphic ($B \simeq C$) if there exist $f, g \in \mathrm{Pol}_1\ \mathcal{A}$ such that $f(B) = C$, $g(C) = B$, $fg = id_C$ and $gf = id_B$.

Theorem 12. *Let $\langle \alpha, \beta \rangle$ be tame. The following are true.*

(1) *$U, V \in M(\alpha, \beta)$ implies $U \simeq V$.*
(2) *$U \in M(\alpha, \beta)$ implies there exists $e \in E$ with $e(A) = U$.*
(3) *$U \in M(\alpha, \beta)$, $f \in \mathrm{Pol}_1\ \mathcal{A}$ and $f(\beta|_U) \nsubseteq \alpha$ implies $f(U) \in M(\alpha, \beta)$ and $f : U \simeq f(U)$.*
(4) *For all $f \in \mathrm{Pol}_1\ \mathcal{A}$, $f(\beta) \nsubseteq \alpha$ implies $f : U \simeq f(U)$ for some $U \in M(\alpha, \beta)$.*

Tame congruence theory classifies the tame quotients of $\mathbf{Con}\ \mathcal{A}$ into five types according to the structure of their minimal sets. Types **1** and **2** are abelian: $\langle \alpha, \beta \rangle$ has one of these types if the algebra $\mathcal{A}/\alpha$ satisfies the $(\beta/\alpha, \beta/\alpha)$-term condition. In terms of the (nonmodular) commutator, this means $[\beta, \beta] \leq \alpha$. Types **3**, **4** and **5** are nonabelian: $\langle \alpha, \beta \rangle$ has one of these types if $[\beta, \beta] \not\leq \alpha$. For our purposes, this distinction between abelian and nonabelian quotients suffices.

We need to describe the structure of minimal sets for prime quotients of nonabelian type. Recall that an $\langle \alpha, \beta \rangle$-*trace* for a minimal set $U \in M(\alpha, \beta)$ is a set of the form $(x/\beta)|_U$ which intersects more than one α-class.

Theorem 13. *If U is an $\langle \alpha, \beta \rangle$-minimal set of type* **3**, **4** *or* **5**, *then it has a unique $\langle \alpha, \beta \rangle$-trace N and there exists $p \in \mathrm{Pol}_2\ \mathcal{A}$ such that*

(1) *$N = \{1\} \cup O$, where $\{1\}$ and O are disjoint $\alpha|_N$-classes,*
(2) *N is closed under p, and $(N/\alpha, p)$ is the two-element meet semilattice,*
(3) *For all $x \in U$, $p(x, 1) = p(1, x) = p(x, x) = x$,*
(4) *For all $x \in U - \{1\}$ and $z \in O$, $p(x, z)\ \alpha\ x\ \alpha\ p(z, x)$,*
(5) *For all $x, y \in U$, $p(x, p(x, y)) = p(x, y)$.*

The polynomial $p(x, y)$ in Theorem 13 is called a *pseudo-meet operation* for U.

Let $\overset{s}{\sim}$ be the equivalence relation on $\mathbf{Con}\ \mathcal{A}$ generated by collapsing all abelian prime quotients, i.e., $\overset{s}{\sim}$ is the reflexive, symmetric, transitive closure of $\{(\alpha, \beta) : \alpha \prec \beta$ and $\mathrm{typ}(\alpha, \beta) \in \{\mathbf{1}, \mathbf{2}\}$. First, we note that $\overset{s}{\sim}$ collapses no more than it is supposed to.

Lemma 14. *For $\gamma \leq \delta$ in* $\mathbf{Con}\ \mathcal{A}$, *we have $\gamma \overset{s}{\sim} \delta$ if and only if for all $\gamma \leq \alpha \prec \beta \leq \gamma$, the quotient $\langle \alpha, \beta \rangle$ is abelian.*

The next result explains our interest in $\overset{s}{\sim}$.

Theorem 15. *For any finite algebra $\mathcal{A}$, the relation $\overset{s}{\sim}$ is a congruence on the lattice* $\mathbf{Con}\ \mathcal{A}$, *and* $\mathbf{Con}\ \mathcal{A}/\overset{s}{\sim}$ *satisfies* $\mathrm{SD}_\wedge$. *If in addition* $\mathbf{5} \notin \mathrm{typ}\{\mathcal{A}\}$, *then* $\mathbf{Con}\ \mathcal{A}/\overset{s}{\sim}$ *satisfies both* $\mathrm{SD}_\wedge$ *and* $\mathrm{SD}_\vee$.

There are corresponding versions of this result for locally finite varieties.

Theorem 16. *Let* $\mathbf{V}$ *be a locally finite variety. Then* $\mathbf{Con}\ \mathcal{A}$ *satisfies* $\mathrm{SD}_\wedge$ *for every $\mathcal{A} \in \mathbf{V}$ if and only if* $\mathrm{typ}\{\mathbf{V}\} \subseteq \{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$.

Theorem 17. *Let* $\mathbf{W}$ *be a locally finite variety. Then* $\mathbf{Con}\ \mathcal{A}$ *satisfies* $\mathrm{SD}_\vee$ *for every finite $\mathcal{A} \in \mathbf{W}$ if and only if* $\mathrm{typ}\{\mathbf{W}\} \subseteq \{\mathbf{3}, \mathbf{4}\}$.

Each of these two types of varieties can be characterized in several other interesting ways, including by Mal'cev conditions (see Chapter 9 of [14]).

4. Congruence lattices of neutral algebras

In this section we will be studying an arbitrarily chosen finite algebra which we denote $\mathcal{A}$. All unspecified references to elements, congruences or operations are references to elements, congruences and operations of the algebra $\mathcal{A}$.

We will quasi-order prime quotients of $\mathbf{Con}\ \mathcal{A}$ in the following way: say that

$$\langle \alpha, \beta \rangle \sqsubseteq \langle \gamma, \delta \rangle$$

if some $\langle \gamma, \delta \rangle$-minimal set contains an $\langle \alpha, \beta \rangle$-minimal set. If some $\langle \gamma, \delta \rangle$-minimal set contains an $\langle \alpha, \beta \rangle$-minimal set, then by Theorem 12(1) and 12(3) every $\langle \gamma, \delta \rangle$-minimal set contains an $\langle \alpha, \beta \rangle$-minimal set. Hence $\sqsubseteq$ is indeed a quasi-order. We will write $\langle \alpha, \beta \rangle \approx \langle \gamma, \delta \rangle$ and say that $\langle \alpha, \beta \rangle$ and $\langle \gamma, \delta \rangle$ are *equivalent* if $\langle \alpha, \beta \rangle \sqsubseteq \langle \gamma, \delta \rangle$ and $\langle \gamma, \delta \rangle \sqsubseteq \langle \alpha, \beta \rangle$. We will write $\langle \alpha, \beta \rangle \sqsubset \langle \gamma, \delta \rangle$ if $\langle \alpha, \beta \rangle \sqsubseteq \langle \gamma, \delta \rangle$, but $\langle \alpha, \beta \rangle \not\approx \langle \gamma, \delta \rangle$. The following two lemmas summarize some elementary properties of the relations $\sqsubseteq$ and $\approx$, respectively.

Lemma 18. *For prime quotients $\langle \alpha, \beta \rangle$ and $\langle \gamma, \delta \rangle$ in $\mathbf{Con}\ \mathcal{A}$,*

(1) *$\langle \alpha, \beta \rangle \sqsubseteq \langle \gamma, \delta \rangle$ holds if and only if $\alpha|_U < \beta|_U$ for some (hence every) $U \in M(\gamma, \delta)$;*

(2) *$\mathrm{Cg}(\gamma, \delta) \leq \mathrm{Cg}(\alpha, \beta)$ in $\mathbf{Con\ Con}\ \mathcal{A}$ implies $\langle \alpha, \beta \rangle \sqsubseteq \langle \gamma, \delta \rangle$.*

Proof. The "only if" part of (1) is obvious. Conversely, let $U \in M(\gamma, \delta)$ and suppose $\alpha|_U < \beta|_U$. By Theorem 12(2), $U = e(\mathcal{A})$ for some idempotent $e \in E$. Now Theorem 12(4) applies to yield that $e : V \simeq e(V)$ for some $V \in M(\alpha, \beta)$. Of course $e(V) \in M(\alpha, \beta)$ and $e(V) \subseteq e(\mathcal{A}) = U$, so $\langle \alpha, \beta \rangle \sqsubseteq \langle \gamma, \delta \rangle$.

To see (2), let $U \in M(\gamma, \delta)$ with say $U = e(\mathcal{A})$ where e is idempotent. The restriction map $\rho : \mathbf{Con}\ \mathcal{A} \twoheadrightarrow \mathbf{Con}\ \mathcal{A}|_U$ is a lattice homomorphism with $\rho(\gamma) = \gamma|_U < \delta|_U = \rho(\delta)$. Hence also $\rho(\alpha) \neq \rho(\beta)$, i.e., $\alpha|_U < \beta|_U$. Thus by (1) we have $\langle \alpha, \beta \rangle \sqsubseteq \langle \gamma, \delta \rangle$. $\square$

Lemma 19. (1) *Perspective prime quotients in $\mathbf{Con}\ \mathcal{A}$ are equivalent.*

(2) *If $\psi \leq \alpha \wedge \gamma$, then the following are equivalent.*

(i) *$\langle \alpha, \beta \rangle \approx \langle \gamma, \delta \rangle$.*

(ii) *$M(\alpha, \beta) = M(\gamma, \delta)$.*

(iii) *$M(\alpha/\psi, \beta/\psi) = M(\gamma/\psi, \delta/\psi)$ in $\mathcal{A}/\psi$.*

(iv) *$\langle \alpha/\psi, \beta/\psi \rangle \approx \langle \gamma/\psi, \delta/\psi \rangle$ in $\mathbf{Con}\ \mathcal{A}/\psi$.*

Proof. (1) is an immediate consequence of Lemma 18(2); it is also a special case of Exercise 2.19 (3) in [14]. (2) is straightforward using the definitions. $\square$

The next lemma provides the crucial technical part of our argument.

Lemma 20. *Assume that $\langle 0, \varphi \rangle$ and $\langle \gamma, \delta \rangle$ are prime quotients in $\mathbf{Con}\ \mathcal{A}$. If $\mathrm{typ}(0, \varphi) \in \{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$ and $\langle 0, \varphi \rangle \approx \langle \gamma, \delta \rangle$, then $\langle 0, \varphi \rangle$ and $\langle \gamma, \delta \rangle$ are perspective.*

Proof. Choose $U \in M(0,\varphi) = M(\gamma,\delta)$. Since $\mathrm{typ}(0,\varphi) \in \{\mathbf{3},\mathbf{4},\mathbf{5}\}$, U has exactly one $\langle 0,\varphi\rangle$-trace and we label it N. By Theorem 13, N has exactly two 0-classes, $I = \{1\}$ and $O = \{0\}$. Let $p(x,y)$ be the pseudo-meet operation of U with respect to $\langle 0,\varphi\rangle$.

Now $U \in M(\gamma,\delta)$, so there exists a pair $(x,y) \in \delta|_U - \gamma|_U$. We would like to show that $(1,0) \in \delta - \gamma$. Let $f \in \mathrm{Pol}_1\, \mathcal{A}$ be given by $f(t) = p(0,t)$. Then $f(0) = 0 = f(1)$, so $f|_U$ is not a permutation, and hence $f(\delta|_U) \subseteq \gamma|_U$. On the other hand, $f(t) = t$ for all $t \in U - \{1\}$. Thus the only possibility for $(x,y) \in \delta|_U - \gamma|_U$ is with either x or $y = 1$; w.l.o.g. say $x = 1$. Then

$$1 = x\ \delta - \gamma\ y = f(y)\ \gamma\ f(x) = p(0,1) = 0.$$

Thus $(1,0) \in \delta$ because $\gamma \le \delta$, while $(1,0) \in \gamma$ would imply $(x,y) \in \gamma$, a contradiction. Therefore we have $(1,0) \in \delta - \gamma$ as claimed.

Now $(0,1) \in \varphi - 0$ and $0 \prec \varphi$, so $\varphi = \mathrm{Cg}(0,1)$. Hence $\varphi \vee \gamma = \mathrm{Cg}(0,1) \vee \gamma = \delta$ and, since $(0,1) \notin \gamma$, $\gamma \wedge \varphi = 0$. Thus the quotients are perspective. $\square$

As a corollary, using Lemma 19, we have a slightly more general assertion.

Corollary 21. *Assume that $\langle \psi,\varphi\rangle$ and $\langle \gamma,\delta\rangle$ are prime quotients in* **Con** $\mathcal{A}$ *with $\psi \le \gamma$. If $\mathrm{typ}(\psi,\varphi) \in \{\mathbf{3},\mathbf{4},\mathbf{5}\}$ and $\langle \psi,\varphi\rangle \approx \langle \gamma,\delta\rangle$, then $\langle \psi,\varphi\rangle$ and $\langle \gamma,\delta\rangle$ are perspective.*

We want to show that the lattice $\mathcal{L} = \mathbf{Con}\ \mathcal{A}/\overset{s}{\sim}$ is upper bounded. To do this, we need to establish the connection between $\sqsubseteq$ and the dual dependence relation D^d on the meet irreducible elements of $\mathcal{L}$.

Lemma 22. *Let $\mathcal{L} = \mathbf{Con}\ \mathcal{A}/\overset{s}{\sim}$. Assume $p,q \in M(\mathcal{L})$ with $q\, D^d\, p$. Let α,γ be the unique maximal members of the $\overset{s}{\sim}$-class corresponding to p,q respectively. Then $\alpha,\gamma \in M(\mathbf{Con}\ \mathcal{A})$ and*

$$\langle \alpha,\alpha^*\rangle \sqsubset \langle \gamma,\gamma^*\rangle.$$

Proof. Clearly α and γ are meet irreducible in **Con** $\mathcal{A}$. Since $q\, D^d\, p$, there exists $x \in L$ such that $q \ge p \wedge x$ and $q \not\ge p^* \wedge x$. Let χ be the maximal congruence in the $\overset{s}{\sim}$-class corresponding to x. It is easy to see that $\gamma \ge \alpha \wedge \chi$ while $\gamma \not\ge \alpha^* \wedge \chi$, and thus $\gamma\, D^d\, \alpha$. This in turn implies $\mathrm{Cg}(\gamma,\gamma^*) \le \mathrm{Cg}(\alpha,\alpha^*)$ in **Con Con** $\mathcal{A}$, and hence $\langle \alpha,\alpha^*\rangle \sqsubseteq \langle \gamma,\gamma^*\rangle$ by Lemma 18(2). It remains only to show that $\langle \alpha,\alpha^*\rangle$ and $\langle \gamma,\gamma^*\rangle$ are not equivalent.

Let $\psi = \alpha \wedge \chi$, and choose $\varphi \in \mathbf{Con}\ \mathcal{A}$ such that $\psi \prec \varphi \le \alpha^* \wedge \chi$. Note that $\langle \psi,\varphi\rangle$ and $\langle \alpha,\alpha^*\rangle$ are perspective, and hence equivalent by Lemma 19(1).

We claim that either $\varphi \le \gamma$ or $\gamma \le \alpha$ must hold. For otherwise, using the meet semidistributivity of $\mathbf{Con}\ \mathcal{A}/\overset{s}{\sim}$, we would have

$$\alpha \wedge \varphi = \psi = \gamma \wedge \varphi \ \overset{s}{\sim}\ \varphi \wedge (\alpha \vee \gamma) = \varphi,$$

a contradiction.

Now suppose that $\langle\alpha,\alpha^*\rangle \approx \langle\gamma,\gamma^*\rangle(\approx \langle\psi,\varphi\rangle)$. If $\varphi \leq \gamma$, then the quotients $\langle\psi,\varphi\rangle$ and $\langle\gamma,\gamma^*\rangle$ violate Corollary 21 since they are equivalent, nonabelian and not perspective. Similarly, if $\gamma \leq \alpha$, then $\langle\gamma,\gamma^*\rangle$ and $\langle\alpha,\alpha^*\rangle$ violate Corollary 21. Hence $\langle\alpha,\alpha^*\rangle \not\approx \langle\gamma,\gamma^*\rangle$, and thus $\langle\alpha,\alpha^*\rangle \sqsubset \langle\gamma,\gamma^*\rangle$. $\square$

It follows immediately from Lemma 22 that **Con** $\mathcal{A}$ contains no D^d-cycle, and hence we have the desired conclusion.

Theorem 23. *For any finite algebra $\mathcal{A}$, the lattice* **Con** $\mathcal{A}/\overset{s}{\sim}$ *is upper bounded.*

Combining this with Theorem 2 and the second part of Theorem 15, we obtain the following.

Theorem 24. *If $\mathcal{A}$ is a finite algebra $\mathcal{A}$ with $\mathbf{5} \notin \mathrm{typ}(\mathcal{A})$, then the lattice* **Con** $\mathcal{A}/\overset{s}{\sim}$ *is both upper and lower bounded.*

Recall that an algebra $\mathcal{A}$ is *neutral* if it satisfies the commutator equation $[\theta,\varphi] = \theta \wedge \varphi$ for all $\theta,\varphi \in$ **Con** $\mathcal{A}$. Thus a finite algebra is neutral if and only if $\overset{s}{\sim}$ is the identity relation on **Con** $\mathcal{A}$, i.e., when every prime quotient of **Con** $\mathcal{A}$ is nonabelian.

Corollary 25. *If $\mathcal{A}$ is a finite neutral algebra, then* **Con** $\mathcal{A}$ *is an upper bounded lattice.*

By virtue of Theorems 16 and 17, applied to each subvariety $\mathrm{V}(\mathcal{A})$ with $\mathcal{A}$ finite, we have the following versions for varieties.

Corollary 26. *If* $\mathbf{V}$ *is a congruence meet semidistributive variety, then the congruence lattices of finite algebras in* $\mathbf{V}$ *are upper bounded.*

Corollary 27. *If* $\mathbf{W}$ *is a congruence join semidistributive variety, then the congruence lattices of finite algebras in* $\mathbf{W}$ *are both upper and lower bounded.*

As an example of Corollary 26, we can consider the variety of *directoids* introduced by J. Ježek and R. Quackenbush in [15]. These are algebras with a single binary operation, denoted by multiplication, satisfying the following equations:

$$\begin{aligned} xx &\approx x \\ (xy)x &\approx xy \\ y(xy) &\approx xy \\ x((xy)z) &\approx (xy)z \ . \end{aligned}$$

It is an elementary exercise to show that a directoid $\mathcal{D}$ can be partially ordered by letting $x \leq y$ iff $xy = x$ (equivalently $yx = x$). With this ordering, the product xy is the smaller of the two elements if they are comparable, and otherwise xy is a common lower bound of x and y. This last property implies that the order on

$\mathcal{D}$ is downward directed. Conversely, any downward directed partially ordered set endowed with a multiplication with these properties yields a directoid.

Using these facts, it is not hard to see that directoids form a variety $\mathbf{D}$ with typ$\{\mathbf{D}\} = \{\mathbf{5}\}$. Consequently, we have the following generalization of the result for semilattices.

Corollary 28. *If $\mathcal{D}$ is a finite directoid, then* $\mathbf{Con}\,\mathcal{D}$ *is an upper bounded lattice.*

A non-congruence-distributive example of Corollary 27 is Polin's variety $\mathbf{P}$ from [27]. This is a finitely generated variety of type $\{\mathbf{3}\}$ (Exercise 9.20(6) of [14]). Thus the congruence lattices of finite algebras in $\mathbf{P}$ are bounded. This was originally proved by Day and Freese in [6]; in fact, they showed that for each $n \in \omega$, $\mathbf{Con}\,\mathcal{F}_{\mathbf{P}}(n)$ is a splitting lattice!

5. Lattices of equational theories

In this section we will show that Theorem 23 has structural consequences for the lattice of subvarieties of any 2-finite variety. These restrictions represent a strengthening of Bill Lampe's "zipper condition" for the 2-finite case. While there have been several refinements of Lampe's condition (see [8], [18]), let us recall the basic version [17].

Theorem 29. *Let* $\mathbf{V}$ *be a variety of algebras. Then the lattice of subvarieties* $\mathcal{L}_v(\mathbf{V})$ *satisfies, for each* $n \geq 2$,

$$(Z_n) \qquad [\&_{0\leq i<n} y \vee x_i \approx z \;\&\; x_0 \wedge \ldots \wedge x_{n-1} \approx 0] \Longrightarrow y \approx z.$$

This can be interpreted as a kind of join semidistributivity at 0. As is often the case, we will find it more convenient to work dually with lattices of equational theories rather than lattices of subvarieties.

If $\mathbf{V}$ is a variety, let $\mathbf{T_V}$ denote the equational theory of $\mathbf{V}$ and let $\mathcal{L}(\mathbf{T_V})$ denote the lattice of equational theories in the language of $\mathbf{V}$ which extend (contain) $\mathbf{T_V}$. Both $\mathbf{V}$ and $\mathbf{T_V}$ are called 2-*finite* if the free algebra $\mathcal{F}_{\mathbf{V}}(2)$ is finite. Any equational theory extending a 2-finite theory is 2-finite. It is well-known that $\mathcal{L}(\mathbf{T_V})$ is isomorphic to the lattice of fully invariant congruences on $\mathcal{F}_{\mathbf{V}}(\omega)$. Therefore, if we expand $\mathcal{F}_{\mathbf{V}}(\omega)$ to an algebra $\mathcal{F}$ on the same universe which has for its basic operations the operations of $\mathcal{F}_{\mathbf{V}}(\omega)$ along with all endomorphisms of $\mathcal{F}_{\mathbf{V}}(\omega)$ as additional unary operations, we will have an algebra with $\mathbf{Con}\,\mathcal{F} \cong \mathcal{L}(\mathcal{T}_{\mathbf{V}})$. It was shown by Ralph McKenzie in [20] that there is a well-defined binary operation $*$ on F which is compatible with the congruences of $\mathcal{F}$. It is defined in the following way. If $\{x_i : i \in \omega\}$ is a free generating set for $\mathcal{F}_{\mathbf{V}}(\omega)$ and $s(x_0, \ldots, x_n)$, $t(x_0, \ldots, x_n) \in F$ where s and t are terms, then

$$s(x_0, \ldots, x_n) * t(x_0, \ldots, x_n) = s(t(x_0, \ldots, x_n), x_1, \ldots, x_n).$$

We will let $\mathcal{F}^*$ denote the algebra $\mathcal{F}$ expanded to include the operation $*$. It is shown in [22] that

$$\mathbf{Con}\ \mathcal{F}^* \cong \mathbf{Con}\ \mathcal{F} \cong \mathcal{L}(\mathbf{T_V}).$$

The most decisive results on the structure of lattices of equational theories have been obtained by studying $\mathcal{F}^*$ and closely related algebras. Observe that for any $s(x_0, \ldots, x_n) \in F$ we have

$$x_0 * s(x_0, \ldots, x_n) = s(x_0, \ldots, x_n) \text{ and } x_1 * s(x_0, \ldots, x_n) = x_1.$$

Thus $*$ is a binary operation on F which has a left unit element and a left absorbing element, *viz.,* x_0 and x_1 respectively. A well-known result from basic commutator theory covers this situation.

Lemma 30. *If $\mathcal{A}$ is an algebra with a binary operation $*$ which has a left zero and left one, then* **Con** $\mathcal{A}$ *satisfies the commutator equation* $[1, \theta] = \theta$*. In particular* $[1, 1] = 1$*, so if $\mathcal{A}$ is finite and $\varphi < 1$, then $(\varphi, 1) \notin \overset{s}{\sim}$.*

Now we are ready to prove our theorem on equational theories. To explain the wording, a lattice homomorphism $h : \mathcal{L} \to \mathcal{K}$ is *1-separating* if $h^{-1} \circ h(1) = \{1\}$.

Theorem 31. *Let* $\mathbf{V}$ *be any 2-finite variety. If $\mathcal{L} = \mathcal{L}(\mathbf{T_V})$ is the lattice of equational theories extending the theory of* $\mathbf{V}$*, then $\mathcal{L}$ has a complete, 1-separating homomorphism onto a finite, upper bounded lattice.*

Proof. We need to show that **Con** $\mathcal{F}^*$ has a complete, 1-separating homomorphism onto a finite, upper bounded lattice. Let e be the endomorphism of $\mathcal{F}_{\mathbf{V}}(\omega)$ determined by $e(x_0) = x_0$ and $e(x_i) = x_1$ for all $i > 0$. Let $A = e(F)$ and define $\mathcal{A} = \mathcal{F}^*|_A$. Notice that A is finite since it is exactly the smallest subuniverse of $\mathcal{F}_{\mathbf{V}}(\omega)$ containing $\{x_0, x_1\}$. The operations of $\mathcal{A}$ are all $f \in \mathrm{Pol}\ \mathcal{A}$ under which A is closed, which includes $*$.

Observe also that since A is the image of an idempotent polynomial of $\mathcal{F}^*$, by Theorem 10 the restriction map

$$|_A : \mathbf{Con}\ \mathcal{F}^* \to \mathbf{Con}\ \mathcal{A}$$

is a complete, onto lattice homomorphism. We claim that this homomorphism is 1-separating. For this, note that any congruence $\theta \in \mathbf{Con}\ \mathcal{F}^*$ is a fully invariant congruence on $\mathcal{F}_{\mathbf{V}}(\omega)$. Hence if $\theta < 1_F$, then $x_0/\theta \neq x_1/\theta$. That is,

$$\theta < 1_F \to x_0/\theta \neq x_1/\theta \to \theta|_A < 1_A \to \theta < 1_F.$$

Thus $\theta < 1_F$ if and only if $\theta|_A < 1_A$, and $|_A$ is 1-separating.

We can finish the proof by showing that **Con** $\mathcal{A}$ has a 1-separating homomorphism onto an upper bounded lattice. For then the composition of this homomorphism with $|_A$ would be a complete, 1-separating homomorphism from **Con** F^* onto a finite, upper bounded lattice. The homomorphism we seek is the natural map

$$\sigma : \mathbf{Con}\,\mathcal{A} \to \mathbf{Con}\,\mathcal{A}/\overset{s}{\sim}.$$

By Theorem 23, this is a homomorphism onto an upper bounded lattice. But $\mathcal{A} = \mathcal{F}^*|_A$ is a finite algebra with a binary operation $*$ which has a left zero and left one, for which case Lemma 30 immediately yields that σ is 1-separating. $\square$

Corollary 32. *Let* $\mathbf{V}$ *be any 2-finite variety. Then the lattice of subvarieties of* $\mathbf{V}$*,* $\mathcal{L}_v(\mathbf{V})$*, has a complete, 0-separating homomorphism onto a finite, lower bounded lattice.*

In the case when $\mathcal{L}_v(\mathbf{V})$ is finite, we can make use of a theorem from the folklore on bounded lattices.

Theorem 33. *If* $\mathcal{L}$ *is a finite lattice, then there exists a least congruence* $\lambda \in \mathbf{Con}\,\mathcal{L}$ *such that* $\mathcal{L}/\lambda$ *is lower bounded.*

This is because finite lower bounded lattices are closed under finite subdirect products. It is useful to have a description of λ. Recall the sets $D_k(\mathcal{L}) \subseteq J(\mathcal{L})$ defined in Section 1. For any finite lattice $\mathcal{L}$, let $D(\mathcal{L})$ be the join subsemilattice of $\mathcal{L}$ generated by $\bigcup_{k\in\omega} D_k(\mathcal{L}) \cup \{0\}$. This is of course a lattice in its own right. Moreover, $D(\mathcal{L})$ satisfies the hypothesis of Theorem 3.1 of [10], so the mapping $h : \mathcal{L} \twoheadrightarrow D(\mathcal{L})$ by

$$h(x) = \bigvee\{u \in D(\mathcal{L}) : u \leq x\}$$

is a lattice homomorphism. Using the arguments in say [16], one can easily show that $\ker h = \lambda$.

Clearly, $\mathcal{L}$ admits a 0-separating homomorphism onto a lower bounded lattice if and only if $0/\lambda = \{0\}$. Using the description of λ given above, this translates as follows.

Theorem 34. *A finite lattice* $\mathcal{L}$ *admits a 0-separating homomorphism onto a lower bounded lattice if and only if every atom of* $\mathcal{L}$ *is in* $D_k(\mathcal{L})$ *for some* $k \in \omega$.

For example, two of the atoms in the lattice of convex subsets of a four-element chain (Figure 1) are not in $D(\mathcal{L})$, and hence this lattice is not isomorphic to $\mathcal{L}_v(\mathbf{V})$ for any 2-finite variety $\mathbf{V}$.

6. Finite lattices of quasivarieties: the locally finite case

An old problem of Garrett Birkhoff [4] asks *which lattices can be represented as the lattice* $\mathcal{L}_q(\mathbf{K})$ *of subquasivarieties of a quasivariety* $\mathbf{K}$ (cf. Mal'cev [19]). Let us note that, for historical reasons, it is customary to talk about *varieties of algebras*, but *quasivarieties of relational systems* (allowing both functions and relational symbols in the type). Surely this distinction is for the most part artificial, but the reader should be aware of it. As an example of where it does make a difference, if we allow infinitely many relational symbols, then the least element 0 of $\mathcal{L}_q(\mathbf{K})$ need not be dually compact.

Viktor Gorbunov, Wiesław Dziobiak and K. V. Adaricheva have made considerable progress in this area, including as a special case the following nice representation theorem from [12] and [2].

Theorem 35. *If* $\mathbf{K}$ *is a locally finite quasivariety with only finitely many relations and* $\mathcal{L}_q(\mathbf{K})$ *is finite, then* $\mathcal{L}_q(\mathbf{K})$ *can be embedded into* $\mathbf{Sub}\,\mathcal{T}$ *for some finite meet semilattice with one,* $\mathcal{T} = \langle T; \wedge, 1\rangle$*. Conversely, for every finite semilattice* $\mathcal{S}$*, there exists a quasivariety* $\mathbf{K}$ *of finite rings such that* $\mathbf{Sub}\,\mathcal{S} \cong \mathcal{L}_q(\mathbf{K})$*.*

Applying Theorem 4, they then obtain the following corollary (which can also be proved directly).

Corollary 36. *If* $\mathbf{K}$ *is a locally finite quasivariety with only finitely many relations and* $\mathcal{L}_q(\mathbf{K})$ *is finite, then* $\mathcal{L}_q(\mathbf{K})$ *is a lower bounded lattice.*

While finite lower bounded lattices do not satisfy any nontrivial lattice equations [9], we do have the following result.

Theorem 37. *For all* $n \geq 2$*, every finite lower bounded lattice satisfies the following quasi-identities, where* $i+1$ *and* $i+2$ *are taken modulo* n*:*
(S_n)
$\&_{0\leq i<n}[x_i \leq x_{i+1} \vee y_i \;\&\; x_i \wedge y_i \leq x_{i+1}] \;\&\; x_0 \wedge \ldots \wedge x_{n-1} \leq y_0 \wedge \ldots \wedge y_{n-1} \Longrightarrow x_0 \approx x_1.$

Proof. If $x_i \leq x_{i+1}$ for all i, then $x_0 = x_1 = x_j$ for all j. So let us suppose $x_i \nleq x_{i+1}$ for some i, and seek a contradiction. Let

$$P = \{a \in J(\mathcal{L}) : a \leq x_j,\ a \nleq x_{j+1} \text{ for some } j\}$$

and let a_0 be an element of minimum D-rank, say r, in P.

Now $a_0 \leq x_j \leq x_{j+1} \vee y_j$ and $a_0 \nleq x_{j+1}$. Also $a_0 \nleq y_j$ else $a_0 \leq x_j \wedge y_j \leq x_{j+1}$. Hence there exists $B \subseteq D_{r-1}(\mathcal{L})$ with $B \ll \{x_{j+1}, y_j\}$ and $a_0 \leq \bigvee B$. Since $a_0 \nleq y_j$, at least one $b_0 \in B$ is not below y_j, and hence below x_{j+1}. Now $b_0 \nleq x_k$ for some k because $\bigwedge x_k \leq y_j$, so for some m we have $b_0 \leq x_m$ and $b_0 \nleq x_{m+1}$. Thus $b_0 \in P$, while the D-rank of b_0 is at most $r-1$, contradicting the choice of a_0. $\square$

Combining this with Corollary 4.3 of [12] (which is a more general version of Theorem 35 above), we obtain another result of Gorbunov.

Theorem 38. *If* $\mathbf{K}$ *is a locally finite quasivariety with only finitely many relations, then* $\mathcal{L}_q(\mathbf{K})$ *satisfies* (S_n) *for every* $n \geq 2$*.*

On the other hand, Gorbunov also shows that these quasi-identities do not hold in arbitrary subquasivariety lattices $\mathcal{L}_q(\mathbf{Q})$; in fact, Tumanov has shown that the quasivariety $\mathbf{Q}$ generated by all lattices of quasivarieties is the class of all lattices satisfying $\mathrm{SD}_\vee$ [29]. Nonetheless, we do not know of any *finite* lattice which is representable as $\mathcal{L}_q(\mathbf{K})$, but not with $\mathbf{K}$ locally finite and of finite type.

Problem. *If $\mathbf{Q}$ is a quasivariety and $\mathcal{L}_q(\mathbf{Q})$ is finite, is $\mathcal{L}_q(\mathbf{Q})$ necessarily lower bounded?*

7. Lattices of quasivarieties: the general case

In this section we will consider lattices $\mathcal{L}_q(\mathbf{K})$ where the quasivariety $\mathbf{K}$ is no longer assumed to be locally finite. Gorbunov and Tumanov have an analogue of Theorem 35 for the general case (see [13], [12]), but they also have examples which show that it will not suffice to yield the results we want. So we will have to resort to other methods to prove the corresponding versions of Corollary 36 and Theorem 38.

Let us recall the closure operator η on $\mathcal{L}_q(\mathbf{K})$ introduced by W. Dziobiak in [7], defined by $\eta(\mathbf{X}) = \mathrm{H}(\mathbf{X}) \cap \mathbf{K}$, where H denotes the closure under homomorphic images. This turns out to be a wonderful tool for investigating the structure of quasivariety lattices. Its properties have been abstracted by Gorbunov and Adaricheva [3]; we say that η is an *equaclosure operator* on the complete lattice $\mathcal{L}$ if it has the following properties.

(1) η is a closure operator on $\mathcal{L}$.
(2) $\eta(0) = 0$.
(3) If $\eta(x) = \eta(y)$ then $\eta(x) = \eta(x \wedge y)$.
(4) $\eta(x) \wedge (y \vee z) = (\eta(x) \wedge y) \vee (\eta(x) \wedge z)$ for all $x, y, z \in \mathcal{L}$.
(5) $\eta(\mathcal{L}) = \{x \in \mathcal{L} : \eta(x) = x\}$ is a dually algebraic lattice in which an element is dually compact iff it is dually compact in $\mathcal{L}$.

Note that (1) and (5) immediately imply a property which we will need later.

(6) If $\{x_i : i \in I\}$ is a chain in $\mathcal{L}$, then $\eta(\bigwedge x_i) = \bigwedge \eta(x_i)$.

(In [2], equaclosure operators are required to satisfy one additional condition, which we do not need here.)

The original use of equaclosure operators is simply that any quasivariety lattice $\mathcal{L}_q(\mathbf{K})$ has Dziobiak's operator naturally defined on it. Thus for example the lattice $\mathbf{Co\ 4}$, which does not admit an equaclosure operator, is not isomorphic to $\mathcal{L}_q(\mathbf{K})$ for any quasivariety $\mathbf{K}$ (see [3]).

Lemma 39. *Let $\mathcal{L}$ be any finite lattice admitting an equaclosure operator η. If $x, y \in J(\mathcal{L})$ and $x \, D \, y$, then $\eta(x) \geq \eta(y)$.*

Proof. Let $z \in L$ be such that $x \leq y \vee z$ but $x \not\leq y_* \vee z$. Then

$$x \leq \eta(x) \wedge (y \vee z) = (\eta(x) \wedge y) \vee (\eta(x) \wedge z) \leq (\eta(x) \wedge y) \vee z$$

whence $\eta(x) \geq y$, and thus $\eta(x) \geq \eta(y)$. $\square$

Theorem 40. *Let $\mathcal{L}$ be any finite lattice admitting an equaclosure operator η. If $x_0\,D\,x_1\,D\,\ldots\,Dx_{n-1}\,D\,x_0$ is a D-cycle in $\mathcal{L}$, then $\bigwedge x_i > 0$.*

Proof. Given a D-cycle in $\mathcal{L}$, by Lemma 39 and property (3) of the definition we obtain $\eta(x_i) = \eta(x_j) = \eta(\bigwedge x_k)$. Since $\eta(z) = 0$ iff $z = 0$, this implies $\bigwedge x_k > 0$. $\square$

If follows from Theorem 40 that in a finite lattice admitting an equaclosure operator, no atom can be in a D-cycle. Thus we can apply Theorem 34.

Corollary 41. *Every finite lattice which admits an equaclosure operator has a 0-separating homomorphism onto a lower bounded lattice.*

Thus the property of having an equaclosure operator implies a sort of lower boundedness at 0. (Adaricheva and Gorbunov observed in [3] that the lattice of convex subsets of the poset in Figure 3 is lower bounded, but does not admit an equaclosure operator. Hence admitting an equaclosure operator is a strictly stronger property than having a 0-separating homomorphism onto a lower bounded lattice.) Obviously it imposes a strong restriction on the structure of quasivariety lattices. On the other hand, it does not eliminate any lattice $\mathcal{L}$ which is an ordinal sum $\mathbf{1}+\mathcal{K}$, where $\mathcal{K}$ is a finite lattice satisfying $\mathrm{SD}_\vee$, from being a quasivariety lattice. Indeed, for comparison one should recall that, despite Lampe's restrictions and McKenzie's additional restrictions in the locally finite case [22], Don Pigozzi has shown that for any dually algebraic lattice $\mathcal{M}$, the ordinal sum $\mathbf{1} + \mathcal{M}$ is isomorphic to $\mathcal{L}_v(\mathbf{W})$ for some variety $\mathbf{W}$ [26].

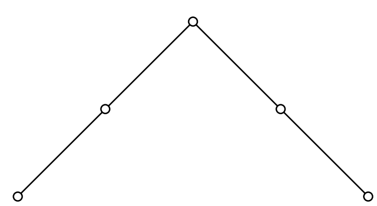

FIGURE 3

Next we want an analogue of Theorem 38 not requiring local finiteness. As in that result, we now allow $\mathcal{L}_q(\mathbf{K})$ to be infinite, but for the next theorem we will need that its least element 0 be dually compact. This will of course be true whenever the type of $\mathbf{K}$ has only finitely many relational symbols.

Theorem 42. *If the least element 0 of $\mathcal{L}_q(\mathbf{K})$ is dually compact, then $\mathcal{L}_q(\mathbf{K})$ satisfies the following quasi-identities for all $n \geq 2$, where $i + 1$ is taken modulo n:*

(T_n) $\quad \&_{0\leq i<n}[x_i \leq x_{i+1} \vee y_i \ \&\ x_i \wedge y_i \leq x_{i+1}] \ \&\ x_0 \wedge \ldots \wedge x_{n-1} \approx 0 \Longrightarrow x_0 \approx 0.$

(Gorbunov proved Theorem 42 under the slightly stronger hypothesis that **K** has finite type [12].)

Proof. Let x_i, y_i ($0 \leq i < n$) satisfy the hypothesis of condition (T_n) in a complete lattice $\mathcal{L}$ which admits an equaclosure operator η and has its 0 element compact. Suppose $x_0 > 0$, and we seek a contradiction. For convenience, we can form infinite cyclically repeating sequences x_i, y_i ($i \in \omega$) with $x_i = x_j$ and $y_i = y_j$ iff $i \equiv j \mod n$. The original sequence satisfies

(i) $x_i \leq x_{i+1} \vee y_i$ for all $i \geq 0$,
(ii) $x_i \wedge y_i \leq x_{i+1}$ for all $i \geq 0$,
(iii) $x_0 > 0$.

We want to transform this sequence, inductively replacing x_i by x'_i, obtaining sequences of the form $\langle x'_0, \ldots, x'_i, x_{i+1}, \ldots \rangle$. Meanwhile, the sequence of y_j's remains fixed. Assume that after $i+1$ steps we have a sequence $\langle x'_0, \ldots, x'_i, x_{i+1}, \ldots \rangle$ satisfying

(1) $x'_i \leq x_{i+1} \vee y_i$ and $x_j \leq x_{j+1} \vee y_j$ for $j > i$,
(2) $x'_i \wedge y_i \leq x_{i+1}$ and $x_j \wedge y_j \leq x_{j+1}$ for $j > i$,
(3) $x'_i > 0$,
(4) $\eta(x'_0) \geq \eta(x'_1) \geq \cdots \geq \eta(x'_i)$,
(5) $j < k \leq i$ and $j \equiv k \mod n$ implies $x'_j \geq x'_k$.

Let $x'_0 = x_0$. Certainly these conditions hold after this one step ($i = 0$).

To get x'_{i+1}, we observe that (1) implies

$$(\dagger) \qquad \begin{aligned} x'_i &\leq \eta(x'_i) \wedge (x_{i+1} \vee y_i) = (\eta(x'_i) \wedge x_{i+1}) \vee (\eta(x'_i) \wedge y_i) \\ &\leq (\eta(x'_i) \wedge x_{i+1}) \vee y_i. \end{aligned}$$

Let $x'_{i+1} = \eta(x'_i) \wedge x_{i+1}$. Since $x'_{i+1} \leq x_{i+1}$, properties (1) and (2) hold with i replaced by $i+1$. Also $x'_{i+1} \leq \eta(x'_i)$ implies $\eta(x'_{i+1}) \leq \eta(x'_i)$, whence (4) holds. Property (4) along with the method of construction yields (5): if $j < k \leq i+1$ and $j \equiv k \mod n$, then $x_j = x_k$, so for $j = 0$ we have $x'_0 = x_0 \geq \eta(x'_{k-1}) \wedge x_k = x'_k$, and for $j > 0$ we have $x'_j = \eta(x'_{j-1}) \wedge x_j \geq \eta(x'_{k-1}) \wedge x_k = x'_k$. To prove (3), suppose $x'_{i+1} = 0$; then using ($\dagger$) and assumption (ii) we get $x'_i \leq x_i \wedge y_i \leq x_{i+1}$, and hence $x'_i \leq \eta(x'_i) \wedge x_{i+1} = x'_{i+1} = 0$, contrary to the induction hypothesis. Thus $x'_{i+1} > 0$.

Now consider the sequence with all primes $\langle x'_0, x'_1, \ldots \rangle$. For $0 \leq k < n$, we form the elements

$$z_k = \bigwedge_{j \equiv k \mod n} x'_j \ .$$

By the dual compactness of 0, $z_k > 0$ for all k. Moreover, we have

$$\eta(z_k) = \bigwedge_{j \equiv k \mod n} \eta(x'_j)$$

and therefore, by property (4), $\eta(z_j) = \eta(z_k) = \bigwedge_{i\geq 0} \eta(x'_i)$ for all j, k. Hence $\eta(z_j) = \eta(\bigwedge z_k) = \eta(0) = 0$ since $\bigwedge z_k \leq \bigwedge x_k = 0$. But this implies $z_j = 0$, which is a contradiction. We conclude that $x_0 = 0$, and in fact by symmetry $x_j = 0$ for all j. $\square$

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