3. Show that the following pairs of abelian groups are not elementarily equivalent.
   (a) \( \mathbb{Z} \) and \( \mathbb{Q} \)
   (b) \( \mathbb{Z} \) and \( \mathbb{Z} \times \mathbb{Z} \)

To show that the above pairs of abelian groups are not elementarily equivalent we will find a first order sentence \( \sigma \) that is true in one structure but false in the other.

**Part a)** Let \( \sigma \) be the sentence \( \forall x \exists y (x = y + y) \). If \( x \in \mathbb{Q} \), then \( \frac{x}{2} \in \mathbb{Q} \) and \( \frac{x}{2} + \frac{x}{2} = x \). Thus \( \mathbb{Q} \models \sigma \). Since \( x + x \) is even for any integer \( x \), no odd integer can equal \( x + x \) for any \( x \in \mathbb{Z} \). Thus, \( \mathbb{Z} \not\models \sigma \).

**Part b)** Now let \( \sigma \) be the sentence \( \exists x \forall y \exists z ((y = x + z + z) \lor (y = z + z)) \). \( \mathbb{Z} \models \sigma \) since if we choose \( x = 1 \), then \( \sigma \) is equivalent to the statement every integer is odd or even. We will show that \( \mathbb{Z} \times \mathbb{Z} \not\models \sigma \).

Assume toward the contrary that \( \mathbb{Z} \times \mathbb{Z} \models \sigma \). Then there exists an element \((a, b) \in \mathbb{Z} \times \mathbb{Z}\) such that for all \((c, d) \in \mathbb{Z} \times \mathbb{Z}\) there exists \((e, f) \in \mathbb{Z} \times \mathbb{Z}\) so that \((c, d) = (a, b) + (e, f) + (e, f) + (e, f)\). In particular, there exists \((e, f) \in \mathbb{Z} \times \mathbb{Z}\) such that \((1, 0) = (a, b) + (e, f) + (e, f)\) or \((1, 0) = (e, f) + (e, f)\). Since \(1 \neq e + e\) for any \( e \in \mathbb{Z} \), we have that \((1, 0) = (a, b) + (e, f) + (e, f)\). Therefore, \(1 = a + e + e\) and hence \(a\) is odd. We also have that there exists \((g, h) \in \mathbb{Z} \times \mathbb{Z}\) such that \((0, 1) = (a, b) + (g, h) + (g, h)\) or \((0, 1) = (g, h) + (g, h)\). Since \(1 \neq h + h\) for any \( h \in \mathbb{Z} \), we have \((0, 1) = (a, b) + (g, h) + (g, h)\). Therefore, \(0 = a + g + g\) and hence \(a\) is even. Since \(a\) cannot be both even and odd, we have our contradiction. Thus \( \mathbb{Z} \times \mathbb{Z} \not\models \sigma \).