4. We wish to show that the class of simple groups, \( \mathcal{K} \), is not elementary in the language of groups, \( \mathcal{L} \). We suppose that \( \mathcal{K} \) is elementary in anticipation of a contradiction – therefore, let \( T \) be an \( \mathcal{L} \)-theory such that \( \mathcal{K} = \{ \mathcal{M} : \mathcal{M} \models T \} \). We use the shorthand notation \( \phi_n(x) \) to denote the \( \mathcal{L} \)-formula \( x \cdot x \cdot (n\text{-times}) \cdot x = e \).

Let \( T' = T \cup \{ \forall x \forall y \ x \cdot y = y \cdot x \} \cup \{ \forall x \neg(x = e) \rightarrow \neg \phi_n(x) : n \in \mathbb{Z}_{\geq 2} \} \).

We claim that any finite subset of \( T' \) has a simple group as a model. Let \( S \) be such a finite subset. If \( S \subset T \cup \{ \forall x \forall y \ x \cdot y = y \cdot x \} \), then \( \mathbb{Z}/2\mathbb{Z} \models S \), and the claim holds. If \( S \not\subset T \cup \{ \forall x \forall y \ x \cdot y = y \cdot x \} \), then by finiteness there exists a maximal integer \( N \) such that the sentence \( \forall x \neg(x = e) \rightarrow \neg \phi_N(x) \) is in \( S \). Let \( p \) denote a prime with \( p > N \) – then \( \mathbb{Z}/p\mathbb{Z} \models S \). In either case, the claim holds.

By compactness, therefore, we have that \( T' \) is satisfiable – however, \( T' \) describes an abelian simple group where every non-identity element has infinite order, and the only abelian simple groups are the cyclic groups of prime order \([1, p. 29]\). This is a contradiction, and thus \( \mathcal{K} \) is not elementary.

References