Math 6000 Model Theory

Trevor Jack, Athena Sparks

Homework 2

5. Look up the definitions of \( \Sigma_1 \) and \( \Pi_1 \) sentences if necessary, and include the definitions in your solution to the following problems.

(a) Show that ultraproducts preserve \( \Sigma_1 \) sentences.
(b) Give an example of a \( \Pi_1 \) sentence not preserved by ultraproducts.

A \( \Sigma_1 \) sentence is a second order sentence of the form

\[ \exists R_1 \ldots \exists R_k \varphi \]

where \( R_1, \ldots, R_k \) are second order variables and \( \varphi \) is a first order formula.

A \( \Pi_1 \) sentence is a second order sentence of the form

\[ \forall R_1 \ldots \forall R_k \varphi \]

where \( R_1, \ldots, R_k \) are second order variables and \( \varphi \) is a first order formula.

**Part a)** Let \( \psi \) be a \( \Sigma_1 \) sentence \( \exists R_1 \ldots \exists R_k \varphi \) in the language \( L \). Then \( \varphi \) can be considered a first order formula in the extended language \( L' = L \cup \{ R_1, \ldots, R_k \} \) where \( R_1, \ldots, R_k \) are now considered as relations, not second order variables.

Let \( I \) be an infinite set and \( \mathcal{U} \) an ultrafilter on \( I \). Assume \( \mathcal{A}_i \) are \( L \)-structures such that \( \mathcal{A}_i \models \psi \) for all \( i \in I \). Extend each \( \mathcal{A}_i \) to the \( L' \)-structure \( \mathcal{A}'_i \) so that \( \mathcal{A}'_i \models \varphi \) for all \( i \). By Loś’s Theorem, we have that \( \prod_{\mathcal{U}} \mathcal{A}'_i \models \varphi \).

Therefore \( \prod_{\mathcal{U}} \mathcal{A}_i \models \psi \). Note that \( \prod_{\mathcal{U}} \mathcal{A}_i \) is an \( L \)-structure and \( \prod_{\mathcal{U}} \mathcal{A}'_i \) is an \( L' \)-structure which was extended by \( \{ R_1, \ldots, R_k \} \). Therefore, we have that \( \prod_{\mathcal{U}} \mathcal{A}_i \models \psi \). Thus ultraproducts preserve \( \Sigma_1 \)-sentences.

**Part b)** Ultraproducts do not preserve all \( \Pi_1 \) sentences. Consider the following example. Let \( \mathcal{U} \) an ultrafilter on \( \omega \). Take \( \mathcal{A} = (\mathbb{N}; \leq) \). Let \( \varphi \) be the \( \Pi_1 \) sentence that says every nonempty subset of \( \mathbb{N} \) has a least element, i.e.

\[ \varphi : \forall S \exists x \forall y (x \in S \land (y \in S \rightarrow x \leq y)) \]
We will show that $\mathbb{A} \models \varphi$ but $\prod_{\mathcal{U}} \mathbb{A} \not\models \varphi$.

Clearly, $\mathbb{A} \models \varphi$. Define $\overline{a_n} \in \mathbb{N}^\omega$ as follows, $\overline{a_n} = (a^i_n)_{i \in \omega}$ where $a^i_n = 0$ if $i \leq n$ and $a^i_n = i - n$ for $i > n$. The collection of these $\overline{a_n}$ for $n \in \omega$ form a subset of $\mathbb{N}^\omega$. For any fixed $k$ we have that $a^i_k = a^i_{k+1}$ for all $i \leq k$ and $a^i_k = a^i_{k+1} + 1$ for all $i > k$. Thus $a^i_k > a^i_{k+1}$ for infinitely many $i$’s. Hence $\overline{a_{k+1}}$ is less than $\overline{a_k}$ in the ultraproduct. Therefore this subset of $\overline{a_n}$’s has no least element. Thus $\prod_{\mathcal{U}} \mathbb{A} \not\models \varphi$. 