Nilpotent Groups of Finite Morley Rank

The main reference for the following is *Groups of Finite Morley Rank* by A. Borovik and A. Nesin. *Simple Groups of Finite Morley Rank*, a book in preparation by T. Altinel, A. Borovik, and G. Cherlin, refines some of the ideas in *Groups of Finite Morley Rank* and serves as a reference for relative definability.

1 Properties of Nilpotent Groups of FMR

We begin with a simple lemma.

**Lemma 1.** Let $G$ be a group of FMR and $H$ be a connected subgroup of $G$. For all $g \in G$, if $[g, H]$ is finite, then in fact $[g, H] = 1$.

**Proof.** The fibers of the commutator map $[g, -] : H \to [g, H]$ are the right cosets of $C_H(g)$ in $H$, so $C_H(g)$ has finite index in $H$. Noting that $C_H(g) = H \cap C_G(g)$ is relatively definable in $H$, the connectedness of $H$ implies that $C_H(g) = H$. \hfill \Box

**Theorem 2.** Let $G$ be a nilpotent group of FMR and $H$ a subgroup of $G$.

(a) If $H$ is infinite and $G$-normal, then $H \cap Z(G)$ is infinite.

(b) If $|G : H|$ is infinite and $H$ is definable, then $|N_G(H) : H|$ is infinite.

**Proof.** First assume that $H$ is infinite and $G$-normal. Choose $i$ minimal such that $A := H \cap Z_{i+1}(G)$ is infinite (here we are using that $G$ is nilpotent). Then $A^\circ$ is also infinite. Additionally, $A^\circ$ is a connected subgroup of $G$ such that $[G, A^\circ] \leq [G, A] \leq H \cap Z_i(G)$. The latter is finite, so by Lemma 1, $[G, A^\circ] = 1$. Thus $A^\circ$ is an infinite subgroup of $H$, central in $G$, so we have proved part (a).

Now assume $|G : H|$ is infinite and that $H$ is definable in $G$ (but $H$ need no longer be infinite or $G$-normal). Let $Z = Z^\circ(G)$, and note that (a) implies that $Z$ is infinite. If $|Z : H \cap Z|$ is infinite, then $|HZ : H|$ is infinite as well. In this case we are done since $H \leq HZ \leq N_G(H)$. Otherwise, $|Z : H \cap Z|$ is finite, so the connectedness of $Z$ implies that $Z \leq H$. We can now proceed by induction on the rank of $G$ to get that $|N_G/Z(H/Z) : H/Z|$ is infinite. Now $N_G/Z(H/Z)/(H/Z) = (N_G(H)/Z)/(H/Z) \cong N_G(H)/H$.

The last statement follows from the fact that if $K$ is the unique subgroup containing $Z$ such that $K/Z = N_G/Z(H/Z)$ then for $k \in K$, $\overline{H}^k = \overline{H}$ implies that $H^k = HZ = H$ (where the bar denotes passage to the quotient $G/Z$). This tells us that $K \subseteq N_G(H)$. The reverse inclusion is clear. \hfill \Box

We now derive other consequences of Lemma 1 including a description of the minimal, infinite, definable subgroups of a group of FMR. We are working to prove the following theorem.
Theorem 3. (Reineke, 1975) In a group of FMR, a minimal, infinite, definable subgroup $A$ is abelian. In fact, $A$ is divisible or an elementary abelian $p$-group.

To prove the second sentence of the theorem, we will need to know about abelian groups of FMR.

Theorem 4. (Macintyre, 1971) Let $G$ be an abelian group of FMR. Then $G = DC$ where $D = T \times N$ and

- $D$ is definable, characteristic, divisible, and connected.
- $C$ is definable, characteristic, and of bounded exponent.
- $T$ is the torsion part of $D$ and is divisible.

Further, $D \cap C$ is finite, and if $G$ is connected, we can take $C$ to be connected.

Proof. Let $G(n) = \{g^n : g \in G\}$, and set $D = \bigcap_{n \in \mathbb{N}} G(n) = \bigcap_{n \in \mathbb{N}} G(n!)$. Then $D$ is a divisible subgroup of $G$, and by DCC, $D = G(n!)$ for some $n$. By a theorem of Baer, stating that in the category of abelian groups divisible groups are injective, $D$ has a complement in $G$, call it $B$. Considering $G/D$, we see that $B$ has bounded exponent of at most $n!$. Set $C = \{g \in G : g^{\exp(B)} = 1\}$. Because $B \leq C$, $G = DC$. Further, $D$ and $C$ are characteristic because they are 0-definable, and divisible groups are connected (they can have no nontrivial, finite quotients and the presence of any nontrivial subgroup of finite index implies the presence of a nontrivial, normal subgroup of finite index).

Setting $T$ to be the torsion part of $D$, it is easily checked that $T$ is divisible. Again by Baer, $T$ has a complement in $D$, call it $N$, which must be torsion free. As $N$ is a quotient of $D$, it is also divisible.

Finally, we show that $D \cap C$ is finite. It will then follow that if $G$ is connected, we have $G = DC^\circ$ (using that $\text{rk} G = \text{rk} D + \text{rk} C^\circ - \text{rk} (D \cap C^\circ)$). We now show that for $k \in \mathbb{N}$, $D$ has only finitely many elements of order dividing $k$ (our proof works for any $k$-divisible, abelian group of FMR). Set $D_k = \{x \in D : x^k = 1\}$. We want to show $D_k$ is finite. If $D_k$ is trivial, we are done. Otherwise, let $y_1 \in D_k \setminus \{1\}$. By the divisibility of $D$, there exists a $y_2 \in D$ such that $y_2^k = y_1$, so $y_2 \in D_{k^2} \setminus D_k$. Repeating, we see that $(D_k)$ is a strictly increasing sequence of definable subgroups. Further, $D_{k^{n+1}} / D_{k^n}$ is isomorphic to $D_k$ via the interpretable map $xD_{k^n} \mapsto x^{k^n}$. If $D_k$ is not finite, $(\text{rk}(D_k^{n+1}))$ is a strictly increasing sequence, which is a contradiction. 

We need two more short lemmas before we reach our goal.

Lemma 5. If $G$ is a group of FMR and $H$ a connected subgroup of $G$ such that $Z(H)$ is finite, then $H/Z(H)$ is a centerless group.

Proof. We wish to show $Z_2(H) = Z(H)$. Let $w \in Z_2(H)$, so that $[w, H] \leq Z(H)$. By Lemma 1, $[w, H] = 1$, so $w$ is central. 

\[ \square \]
Lemma 6. If $G$ is a connected group of FMR such that $C_G(x)$ is finite for all $x \in G \setminus \{1\}$, then $G = 1$.

Proof. If $G$ is finite, the connectedness of $G$ implies that $G = 1$. By way of contradiction, assume that $G$ is infinite. We know that $G$ is the disjoint union of $\{1\}$ and the nontrivial conjugacy classes of $G$. However, for all $x \in G \setminus \{1\}$, the right coset space $G/C_G(x)$ is in interpretable bijection with $x^G$. Thus $\text{rk}(x^G) = \text{rk}(G) - \text{rk}(C_G(x)) = \text{rk}(G)$. Since $G$ is degree 1, $G$ must have only one nontrivial conjugacy class. Hence, for any $x \in G \setminus \{1\}$, $G = x^G \cup \{1\}$. Further, $x \in C_G(x)$ which is finite, so $x$ has finite order. Thus every nontrivial element of $G$ has the same finite order, so $\text{exp}(G) = p$ for some prime $p$.

Now, $N_G(\langle x \rangle)/C_G(x)$ is a finite group with order dividing $|\text{Aut}(\langle x \rangle)| = p - 1$, but it certainly must also have exponent dividing $p$ (so $p$ divides the order of $N_G(\langle x \rangle)/C_G(x)$). We conclude that $N_G(\langle x \rangle)/C_G(x)$ is trivial, so $N_G(\langle x \rangle)$ acts trivially on $\langle x \rangle$ (this also follows from the fact that the only definable action of a connected group of FMR on a finite set is the trivial action). If $\text{exp}(G) > 2$, we contradict the fact that $x^2 \in x^G$. If $\text{exp}(G) = 2$, $G$ is abelian, and we contradict the fact that $C_G(x)$ is finite. \hfill $\square$

We are now able to prove the theorem of Reineke that we have been working towards.

Proof of Theorem 3. We wish to show that $Z(A) = A$. By the minimality of $A$, it is enough to show that $Z(A)$ is infinite. Note that $A$ is connected. Towards a contradiction, assume $Z(A)$ is finite. By Lemma 5, $\overline{A} := A/Z(A)$ is centerless, and (by assumption) $\overline{A}$ has no proper infinite, definable subgroups. Thus for all $\pi \in \overline{A}$, we have that $C_{\overline{A}}(\pi)$ is finite. Since $\overline{A}$ is connected, this forces $\overline{A}$ to be trivial, contradicting the fact that $A$ is infinite. Thus $A$ is abelian.

By the Macintyre’s theorem, $A = D \ast C$ where $D$ is divisible, $C$ has bounded exponent, and both are definable (in $A$ hence in $G$). Certainly $D$ or $C$ must be infinite, so by the minimality of $A$ the one that is infinite must equal $A$. If $A = D$, we are done. Otherwise, $A$ has bounded exponent, say $\text{exp}(A) = n$. Let $p$ be a prime dividing $n$. Then $\varphi : A \longrightarrow A : a \mapsto a^p$ is an endomorphism whose image has exponent $n/p$, so the image is a proper definable subgroup (of $A$ hence of $G$). By the minimality of $A$, the image of $\varphi$ must be finite, so the kernel of $\varphi$ is infinite (hence equal to $A$). \hfill $\square$
2 The Structure of Nilpotent Groups of FMR

Our goal is to prove the following theorem which will come in two pieces.

**Theorem 7. (Nesin, 1991)** Let $G$ be a nilpotent group of FMR. Then $G = D \ast C$ where $D = T \times N$ and

- $D$ is definable, characteristic, divisible, and connected.
- $C$ is definable, characteristic, and of bounded exponent.
- $T$ is the torsion part of $D$ and is divisible and central in $G$.
- $N$ is torsion free.

Further, $D \cap C$ is central and finite, and if $G$ is connected, we can take $C$ to be connected.

We begin by showing that we can decompose $G$ as a central product of a divisible subgroup and a subgroup of bounded exponent. We will later address the decomposition for $D$.

**Lemma 8.** Let $G$ be a nilpotent group of FMR. Then $G = D \ast C$ where

- $D$ is definable, characteristic, divisible, and connected.
- $C$ is definable, characteristic, and of bounded exponent.

Further, $D \cap C$ is central and finite, and if $G$ is connected, we can take $C$ to be connected.

**Proof.** We begin by explaining why it is enough to show that $G = DB$ for $D$ a divisible subgroup and $B$ a subgroup of finite exponent. Note that $D$ centralizes $B$ (see background). Let $n = \exp(B)$. For $g \in G$, we may write $g = db$ for $d \in D$ and $b \in B$, and $g^n = d^n b^n = d^n$ (since $D$ centralizes $B$).

Thus, $D = \{d^n : d \in D\} = \{g^n : g \in G\}$, so $D$ is in fact 0-definable and characteristic. Now set $C = \{g \in G : g^n = 1\}$. It is not clear that $C$ is a subgroup, but $C$ is 0-definable, hence characteristic, set containing $B$. To show $C$ is a subgroup we need only show that $C$ is closed under multiplication (inversion and 1 are clear). Using our previous observations $C = \{db \in G : d \in D, b \in B, d^n = 1\} \subseteq \{db \in G : d \in D, b \in B, d \text{ is central in } G\}$ (see background). Thus for $c_1, c_2 \in C$, $(c_1 c_2)^n = d_{c_1}^n d_{c_2}^n (b_{c_1} b_{c_2})^n = 1$, so $C$ is a subgroup. Clearly $G = DC$ where $D$ centralizes $C$, so $G = D \ast C$. Finally, $D \cap C$ is contained in Tor($D$) which we have already mentioned is central in $G$. Tor($D$) is easily seen to be divisible, so the argument in Theorem 4 (Macintyre’s Theorem) shows that Tor($D$) has only finitely many elements of order dividing $n$. Hence, $D \cap C$ is finite. Thus, it suffices to show that $G = DB$ for $D$ a divisible subgroup and $B$ a subgroup of finite exponent.

Now suppose that the lemma is not true, and let $G$ be a counterexample of minimal rank and degree. We will show that in fact $G = DB$ for $D$ a divisible subgroup and $B$ a subgroup of finite exponent, which will be our contradiction. Now, $G$ is not abelian (by Macintyre’s Theorem). Set $Z = Z^o(G)$, and (by minimality) write $Z = D_0 C_0$ with $D_0$ and $C_0$ as in the
lemma with both connected. Note that $D_0$ and $C_0$ are central subgroups of $G$ and are actually characteristic in $G$. We now consider two cases.

**Case 1:** Assume that $C_0 \neq 1$. Thus, the connectedness of $C_0$ implies that $C_0$ is infinite, so $\text{rk}(G/C_0) < \text{rk} G$. By induction, $G/C_0 = (D_1/C_0)*(C_1/C_0)$ where $(D_1/C_0)$ and $(C_1/C_0)$ are as expected. Note that, as $C_0$ is central, $C_1$ is of bounded exponent and we have the following lattice.

$$\begin{array}{c}
\text{G} \\
\downarrow \\
D_1 \\
\downarrow \\
D_1 \cap C_1 \\
\downarrow \\
C_0
\end{array}$$

If $D_1 \neq G$, then we may write $D_1 = D_2 * C_2$ for $D_2$ divisible and $C_2$ of bounded exponent. Then $G = D_1 C_1 = D_2(C_2 C_1)$. Since $C_2 C_1$ is nilpotent and each factor is of bounded exponent, $C_2 C_1$ is of bounded exponent (see background), so we are done with $D = D_2$ and $B = C_2 C_1$.

Next consider when $D_1 = G$. First note that $G/C_0$ is divisible. We now work to produce a proper, definable subgroup $H$ of $G$ such that $G = H C_0 = H \ast C_0$ (noting that $C_0$ is central). For then, by minimality, $H = D_3 \ast C_3$, and we are done with $D = D_3$ and $B = C_3 C_0$. Let $n = \text{exp}(C_0)$, and set $X = \{g^n : g \in G\}$ and $H = \langle X \rangle$. Since $G/C_0$ is $n$-divisible, $G = X C_0$. We show $X$ is $n$-divisible. For $x = g^n$ in $X$, write $g = yc$ for $y \in X$ and $c \in C_0$. Since $C_0$ is central, $x = g^n = y^n c^n = y^n$, so $X$ is $n$-divisible. Now, $H$ is nilpotent and $H/H'$ is $n$-divisible, as it is an abelian group generated by the $n$ divisible set $XH'$, so $H$ is $n$-divisible (see background). Thus $H = X$ is a 0-definable, characteristic subgroup. As before, $H \cap C_0$ is contained in Tor ($H$) which is central in $G$. Tor ($H$) is easily seen to be $n$-divisible, so Tor ($H$) has only finitely many elements of order dividing $n$. Hence, $H \cap C_0$ is finite. A rank argument shows that $\text{rk} (H) < \text{rk} (G)$, so $H$ is proper. This finishes case 1.

**Case 2:** Assume that $C_0 = 1$. Then $D_0$ is infinite, so $\text{rk}(G/D_0) < \text{rk} G$. By induction, $G/D_0 = (D_1/D_0) \ast (C_1/D_0)$ where $(D_1/D_0)$ and $(C_1/D_0)$ are as expected. Note that, as $D_0$ is central, $D_1$ is divisible and we have the previous lattice with $D_0$ replacing $C_0$.

If $C_1 \neq G$, then we may write $C_1 = D_2 * C_2$ for $D_2$ divisible and $C_2$ of bounded exponent. Then $G = D_1 C_1 = (D_1 D_2) C_1$. Since $D_1 D_2$ is nilpotent and each factor is divisible, $D_1 D_2$ is divisible (see background), so we are done with $D = D_1 D_2$ and $B = C_1$. 
Next consider when \( C_1 = G \). This time \( G/D_0 \) is of bounded exponent. Let \( n = \exp(G/D_0) \), and set \( X = \{ g \in G : g^n = 1 \} \) and \( B = \langle X \rangle \). We show \( G = BD_0 \). Let \( g \in G \). Then \( g^n \in D_0 \), so the divisibility of \( D_0 \) implies that there is a \( d \in D_0 \) such that \( g^n = d^n \). Because \( D_0 \) is central, \( (gd^{-1})^n = 1 \), so \( gd^{-1} \in B \). Thus, \( g = bd \) for some \( b \in B \), so \( G = BD_0 \). Now, \( B \) is nilpotent and \( B/B' \) is of bounded exponent, as it is an abelian group generated by \( XB' \), so \( B \) is of bounded exponent (see background). Thus, we are done with \( D = D_0 \), and this finishes case 2.

Now we address the decomposition for \( D \) in Theorem 7.

**Lemma 9.** Let \( D \) be a divisible nilpotent group of FMR, and \( T = \text{Tor}(D) \). Then \( T \) is a divisible central subgroup of \( D \), and \( D = T \times N \) for some torsion free divisible subgroup \( N \).

**Proof.** See *Groups of Finite Morley Rank* by A. Borovik and A. Nesin. \( \square \)