The Math in "Laser Light Math"

When graphed, many mathematical curves are beautiful to view. These curves are usually brought into graphic form by incorporating such devices as a plotter, printer, video screen, or mechanical spirograph tool. While these techniques work, and can produce interesting images, the images are normally small and not animated. To create large-scale animated images, such as those encountered in the entertainment industry, light shows, or art installations, one must call upon some unusual graphing strategies. It was this desire to create large and animated images of certain mathematical curves that led to our design and implementation of the Laser Light Math projection system. In our interdisciplinary efforts to create a new way to graph certain mathematical curves with laser light, Professor Lessley designed the hardware and software and Professor Beale constructed a simplified set of harmonic equations.

Of special interest was the issue of graphing a family of mathematical curves in the roulette or spirograph domain with laser light. Consistent with the techniques of making roulette patterns, images created by the Laser Light Math system are constructed by mixing sine and cosine functions together at various frequencies, shapes, and amplitudes. Images created in this fashion find birth in the mathematical process of making "roulette" or "spirograph" curves. From your childhood, you might recall working with a spirograph toy to which you placed one geared wheel within the circumference of another larger geared wheel. After inserting your pen, then rotating the smaller gear around or within the circumference of the larger gear, a graphed representation emerged of a certain mathematical curve in the roulette family (such as the epitrochoid, hypotrochoid, epicycloid, hypocycloid, or perhaps the beautiful rose family). The spirograph, of course, is simply a clever mechanical plotting device. Performing this same artistry with a laser beam and its tiny dot of light is, however, a more demanding process.

The Problem: Moving from the Spirograph to the Laser

Herein lies the problem: the spirograph, as a graphing device, utilizes a geared wheel running inside or outside of another larger geared wheel. The larger wheel does not rotate. Having the larger wheel remain stationary will not work for laser projection devices. To form a complete projected pattern(s) of any mathematical curve with a small laser "dot" requires a modified and dynamic approach. To do this, you must "scan" the laser dot rapidly through a complete image path at least sixteen times a second before the image will appear solid. This effect is known as "persistence of vision."

For a better understanding of this idea, take a hand-held laser pointer and move it back and forth faster than sixteen times per second. This action will create the appearance of a horizontal line. Similarly, moving the pointer rapidly in a circular pattern will create the appearance of a solid circle in light. In reality, what you create by moving a dot of laser light quickly is the "illusion" of a solid line or pattern (thanks to the persistence of vision effect). Essentially, this is the secret to creating any laser image. Of course, it is important to remember that instead of moving the laser itself you must use some "hardware" to "reflect" the laser dot rapidly through its defined pattern. More on the hardware will follow later.

If you use the laser pointer to draw a series of smaller circles that follow the path of a larger circle (a roulette shape), you will discover that--unlike the mechanical spirograph--the trace and the base circles rotate simultaneously. Additionally, it will become clear that both circles can possess individual frequency and diameter factors.

For an informative historical example of this idea, think back to your basic astronomy class and the lectures on Ptolemy and how he tried to explain the visual motion of the planets by developing the idea of "epicycles." A full discussion of Ptolemy's ideas is beyond the scope of this paper, but of critical importance is how he demonstrated the idea of creating roulette patterns in a slightly different manner than with our conventional spirograph. To be exact, his approach (that of "epicycles") has our trace circle rotating "on the circumference" of the base circle. And, especially important, both circles can maintain differing rotational speeds (frequencies) and rotational directions.

Figure 1 compares Ptolemy's epicycle approach to the traditional spirograph technique.

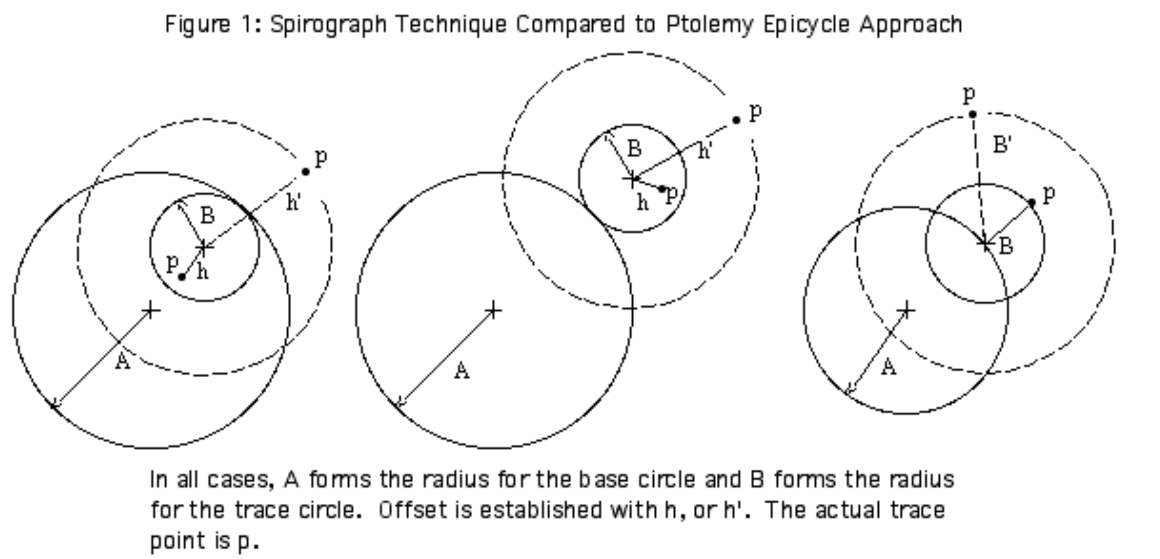

Figure 1: *Traditional Spirograph Device Compared to Ptolemy's Epicycle Approach*

The first two drawings in Figure 1 follow the normal graphing strategy used by a spirograph device. The third element shows how the tracing strategy in Laser Light Math follows Ptolemy's idea and how it differs from the normal spirograph method (the trace circle runs on the circumference of the base circle).

After embracing the idea of using an epicycle approach, we began to form the math required to make this all work within the context of a laser projection system where digital and analog devices must co-exist. The usual equations for making roulette or spirograph curves are modified to incorporate the epicycle approach where two circles can rotate at differing frequencies, sometimes in differing directions, and often with differing diameters. Additionally, the revised

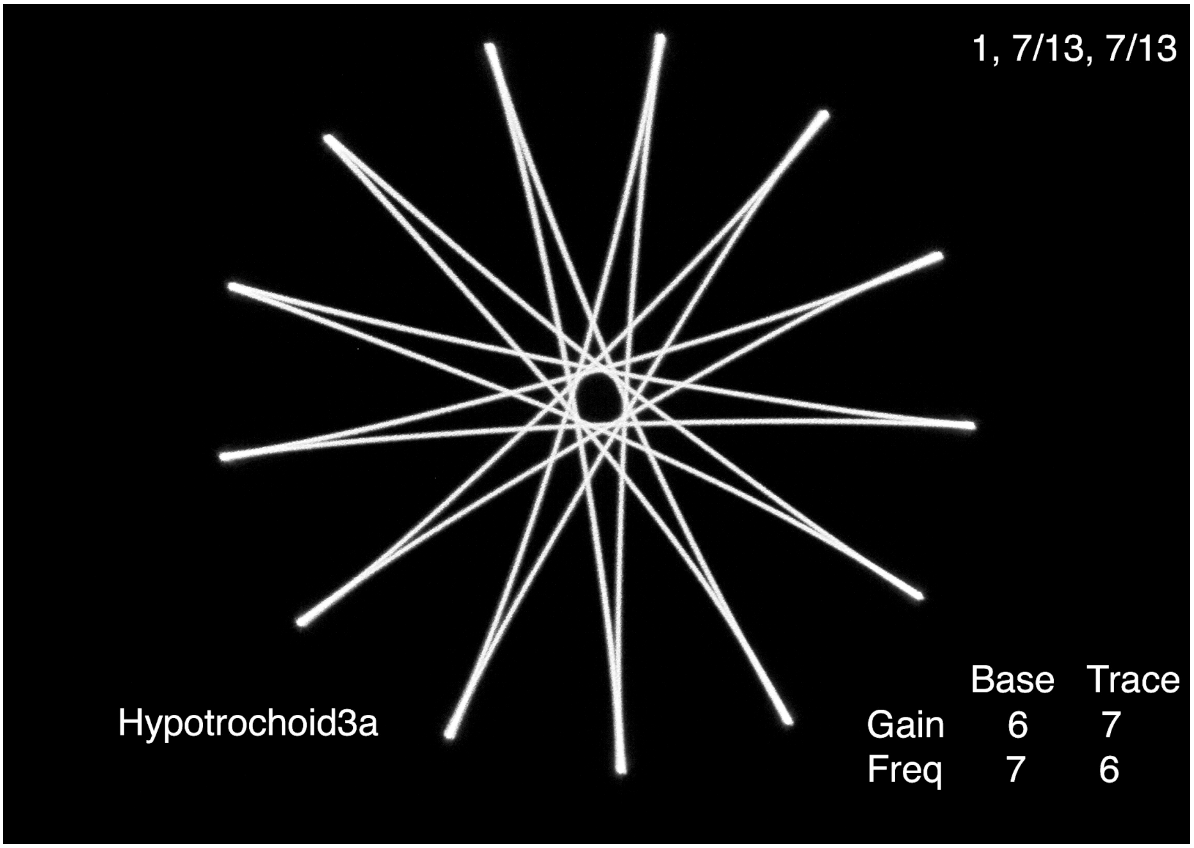

Figure2: *A Typical Hypotrochoid Image*

equations carefully engage the idea of "offset" in which there are times when the laser dot does not ride on the circumference of the trace circle: instead, it can be inside or outside of that circumference. By incorporating offset into the epicycle strategy, we can create a multitude of striking images from the epitrochoid and hypotrochoid curve families.

Figure 2 and Figure 3 are examples of curves created with the epicycle approach. Both images were created with the Laser Light Math projection system. To see more examples, just go to the home page of this site where you can find links to color photos, moving images, and additional information regarding the Laser Light Math project.

Creating the laser image in Figure 2 requires a mathematical approach similar to the traditional graphing of a hypotrochoid curve. The equation, however, must be modified to accommodate the "dynamic" nature of the scanning (base and trace circles). The traditional parametric equation for graphing a hypotrochoid curve is:

$$x = (a-b)\cos(t) + h\cos\left((\frac{a}{b}-1)t\right)$$
$$y = (a-b)\sin(t) - h\sin\left((\frac{a}{b}-1)t\right).$$

In our approach, however, we use base and trace oscillators to form the images so the equations become

$$x = (a-b)\cos(\omega_0 t) + h\cos\left(\frac{a-b}{b}\omega_0 t\right)$$
$$y = (a-b)\sin(\omega_0 t) - h\sin\left(\frac{a-b}{b}\omega_0 t\right),$$

where $\omega_0 = 2\pi f_{base}$. The base frequency f_{base} is the number of times per second that the base oscillator completes a cycle.

In Figure 3, which is from the epitrochoid family, the numbers at the top of the laser image photo represent the regular graphing variables, and the numbers in the lower right hand corner represent the modified values that are applied to various control faders within our projection system. Translating the traditional values into required revised values usually requires only simple addition and subtraction.

In creating the Figure 3 epitrochoid image we use the following parametric equations for our x and y data input:

$$x = (a+b)\cos(\omega_0 t) - h\cos\left(\frac{a+b}{b}\omega_0 t\right)$$
$$y = (a+b)\sin(\omega_0 t) - h\sin\left(\frac{a+b}{b}\omega_0 t\right).$$

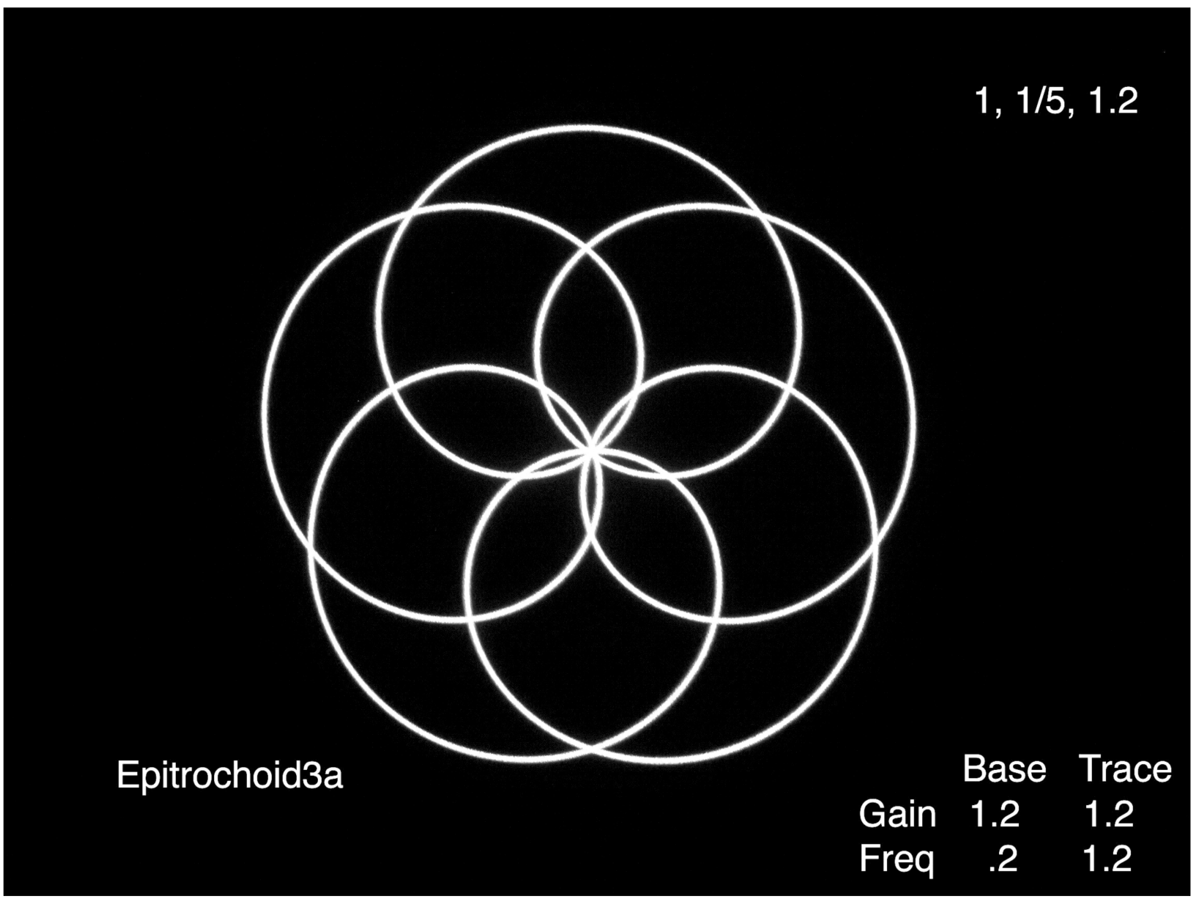

Figure 3: *A Typical Epitrochoid Image*

Please note that when graphing epitrochoid or hypotrochoid functions with a mechanical spirograph plotter, you usually incorporate a specific offset value to the trace point in association with the trace circle. However, in the case of our epicycle approach, where the trace circle's center always follows the path of the base circle's circumference (the Ptolemy epicycle idea), the effect of offset is accomplished through varying the diameter ratios of the two circles and the rotational speeds of each circle.

Perhaps the most fascinating aspect of these laser images is the manner in which they evolve with time. If the two frequencies are incommensurate ($\frac{a}{b} \neq \frac{p}{q}$ where p and q are integers, i.e. a/b is not a fraction but rather an irrational number), then the various colors in the image slowly rotate with respect to each other. Dramatic changes in the colors in the image occur periodically when the rotating patterns overlap. This is caused by the spectral additions of colors and their interaction with human color perception. Additionally, in Laser Light Math, when two traces converge or overlap, a striking effect occurs where it appears as if the image suddenly pauses. This "illusion of pause" probably relates to the overlap of Gaussian beams. Such overlap causes the combined beam to appear not only more intense and wider but also temporarily stationary; and the color we observe at the overlap point is the additive value of the two trace lines. A red beam overlapping a green beam creates a yellow composite.

Laser Light Math does virtually all of this work within the frame of epitrochoid and hypotrochoid functions. This practice does not necessarily limit the range of images, however, since a number of "special cases" exist that can be generated within this basic framework. The special cases are:

hypotrochoid:
- hypocycloid: h=b
- ellipse: a=2*b
- rose: a=(2n)*h/(n+1), b=(n-1)/(n+1)*h where "n" is the number of petals

epitrochoid:
- epicycloid: h=b
- limacon: a=b
- circle: a=0

In terms of comprehending how Laser Light Math works, it is quite important to understand one essential idea in relationship to sine and cosine math: in a sine/cosine relationship, in which two waves are both sinusoidal in shape, you can create a circle when the x and y waveforms have the same frequency and are exactly ninety degrees out of phase. In electronics, this is referred to as a "quadrature" relationship. Oscillations are in quadrature relationship when they are separated in phase by 90° (π/ 2 radians, or ¼ of a period). The wave shapes produced in Laser Light Math are heavily dependent upon maintaining a quadrature relationship.

To gain greater artistic flexibility, we can also mix other waveforms in quadrature (saw, triangle, square, saw tooth for example) inside of sinusoidal waves. Fourier analysis shows that any periodic function can be constructed by a sum of sines and cosines. Some unusual effects can be generated in this manner. By mixing two square waves of equal frequency and amplitude in a quadrature relationship, the system produces a nicely defined square. Similarly, two triangle waves in quadrature relationship (one on the X axis and the other on the Y axis) will produce a diamond image. Thus, by using sinusoidal waves, square waves, triangle waves, and saw tooth waves, it is possible to create a diverse range of unusual and beautiful images that cannot be generated through the traditional spirograph plotter technique (which normally uses only sinusoidal functions). By possessing this increased image making capacity, the Laser Light Math system opens up new image-making possibilities for laser artists.