# CSCI5254: Convex Optimization & Its Applications

## Convex optimization problems

- optimization problem in standard form
- convex optimization problems
- quasiconvex optimization
- linear optimization
- quadratic optimization
- generalized inequality constraints
- semidefinite programming
- geometric programming
- vector optimization

# Optimization problem in standard form

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{array}
$$

- $x \in \mathbf{R}^n$ is the optimization variable
- $f_0 : \mathbf{R}^n \to \mathbf{R}$ is the objective or cost function
- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$, are the inequality constraint functions
- $h_i : \mathbf{R}^n \to \mathbf{R}$ are the equality constraint functions

optimal value:

$$
p^\star = \inf\{f_0(x) \mid f_i(x) \leq 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p\}
$$

- $p^\star = \infty$ if problem is infeasible (no $x$ satisfies the constraints)
- $p^\star = -\infty$ if problem is unbounded below

# Optimal and locally optimal points

$x$ is feasible if $x \in \mathbf{dom}\, f_0$ and it satisfies the constraints

a feasible $x$ is optimal if $f_0(x) = p^\star$; $X_{\rm opt}$ is the set of optimal points

$x$ is locally optimal if there is an $R > 0$ such that $x$ is optimal for

$$\begin{array}{ll} \mbox{minimize (over } z) & f_0(z) \\ \mbox{subject to} & f_i(z) \leq 0, \quad i = 1, \ldots, m, \quad h_i(z) = 0, \quad i = 1, \ldots, p \\ & \|z - x\|_2 \leq R \end{array}$$

examples (with $n = 1$, $m = p = 0$)

- $f_0(x) = 1/x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = 0$, no optimal point
- $f_0(x) = -\log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = -\infty$
- $f_0(x) = x \log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = -1/e$, $x = 1/e$ is optimal
- $f_0(x) = x^3 - 3x$, $p^\star = -\infty$, local optimum at $x = 1$

# Implicit constraints

the standard form optimization problem has an implicit constraint

$$x \in \mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i \;\cap\; \bigcap_{i=1}^{p} \mathbf{dom}\, h_i,$$

- we call $\mathcal{D}$ the domain of the problem
- the constraints $f_i(x) \leq 0$, $h_i(x) = 0$ are the explicit constraints
- a problem is unconstrained if it has no explicit constraints ($m = p = 0$)

example:

$$\text{minimize} \quad f_0(x) = -\textstyle\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints $a_i^T x < b_i$

# Feasibility problem

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

can be considered a special case of the general problem with $f_0(x) = 0$:

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- $p^\star = 0$ if constraints are feasible; any feasible $x$ is optimal
- $p^\star = \infty$ if constraints are infeasible

# Convex optimization problem

standard form convex optimization problem

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& a_i^T x = b_i, \quad i = 1, \ldots, p
\end{array}
$$

- $f_0$, $f_1$, . . . , $f_m$ are convex; equality constraints are affine
- problem is *quasiconvex* if $f_0$ is quasiconvex (and $f_1$, . . . , $f_m$ convex)

often written as

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

important property: feasible set of a convex optimization problem is convex

## example

$$
\begin{array}{ll}
\text{minimize} & f_0(x) = x_1^2 + x_2^2 \\
\text{subject to} & f_1(x) = x_1/(1 + x_2^2) \leq 0 \\
& h_1(x) = (x_1 + x_2)^2 = 0
\end{array}
$$

- $f_0$ is convex; feasible set $\{(x_1, x_2) \mid x_1 = -x_2 \leq 0\}$ is convex
- not a convex problem (according to our definition): $f_1$ is not convex, $h_1$ is not affine
- equivalent (but not identical) to the convex problem

$$
\begin{array}{ll}
\text{minimize} & x_1^2 + x_2^2 \\
\text{subject to} & x_1 \leq 0 \\
& x_1 + x_2 = 0
\end{array}
$$

# Local and global optima

any locally optimal point of a convex problem is (globally) optimal

proof: suppose $x$ is locally optimal and $y$ is optimal with $f_0(y) < f_0(x)$

$x$ locally optimal means there is an $R > 0$ such that

$$z \text{ feasible}, \quad \|z - x\|_2 \leq R \quad \Longrightarrow \quad f_0(z) \geq f_0(x)$$

consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$

- $\|y - x\|_2 > R$, so $0 < \theta < 1/2$
- $z$ is a convex combination of two feasible points, hence also feasible
- $\|z - x\|_2 = R/2$ and

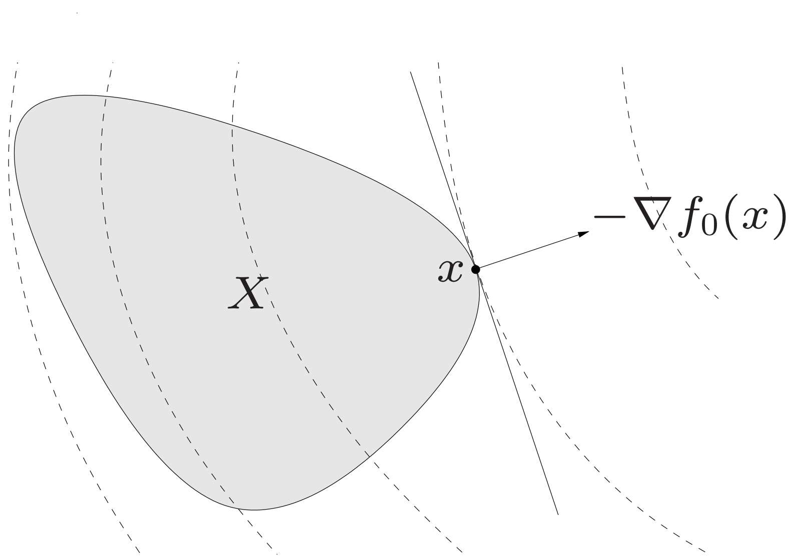
$$f_0(z) \leq \theta f_0(x) + (1 - \theta) f_0(y) < f_0(x)$$

which contradicts our assumption that $x$ is locally optimal

# Optimality criterion for differentiable $f_0$

$x$ is optimal if and only if it is feasible and

$$\nabla f_0(x)^T(y - x) \geq 0 \quad \text{for all feasible } y$$




if nonzero, $\nabla f_0(x)$ defines a supporting hyperplane to feasible set $X$ at $x$

- unconstrained problem: $x$ is optimal if and only if

$$x \in \mathbf{dom}\, f_0, \qquad \nabla f_0(x) = 0$$

- equality constrained problem

$$\text{minimize} \quad f_0(x) \quad \text{subject to} \quad Ax = b$$

$x$ is optimal if and only if there exists a $\nu$ such that

$$x \in \mathbf{dom}\, f_0, \qquad Ax = b, \qquad \nabla f_0(x) + A^T\nu = 0$$

- minimization over nonnegative orthant

$$\text{minimize} \quad f_0(x) \quad \text{subject to} \quad x \succeq 0$$

$x$ is optimal if and only if

$$x \in \mathbf{dom}\, f_0, \qquad x \succeq 0, \qquad \left\{ \begin{array}{ll} \nabla f_0(x)_i \geq 0 & x_i = 0 \\ \nabla f_0(x)_i = 0 & x_i > 0 \end{array} \right.$$

# Equivalent convex problems

two problems are (informally) equivalent if the solution of one is readily obtained from the solution of the other, and vice-versa

some common transformations that preserve convexity:

- eliminating equality constraints

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

  is equivalent to

$$
\begin{array}{ll}
\text{minimize (over } z\text{)} & f_0(Fz + x_0) \\
\text{subject to} & f_i(Fz + x_0) \leq 0, \quad i = 1, \ldots, m
\end{array}
$$

  where $F$ and $x_0$ are such that

$$
Ax = b \quad \Longleftrightarrow \quad x = Fz + x_0 \text{ for some } z
$$

- introducing equality constraints

$$
\begin{array}{ll}
\text{minimize} & f_0(A_0x + b_0) \\
\text{subject to} & f_i(A_ix + b_i) \leq 0, \quad i = 1, \ldots, m
\end{array}
$$

is equivalent to

$$
\begin{array}{ll}
\text{minimize (over } x,\ y_i) & f_0(y_0) \\
\text{subject to} & f_i(y_i) \leq 0, \quad i = 1, \ldots, m \\
 & y_i = A_ix + b_i, \quad i = 0, 1, \ldots, m
\end{array}
$$

- introducing slack variables for linear inequalities

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m
\end{array}
$$

is equivalent to

$$
\begin{array}{ll}
\text{minimize (over } x,\ s) & f_0(x) \\
\text{subject to} & a_i^T x + s_i = b_i, \quad i = 1, \ldots, m \\
 & s_i \geq 0, \quad i = 1, \ldots m
\end{array}
$$

- epigraph form: standard form convex problem is equivalent to

$$\begin{array}{ll} \text{minimize (over } x, t) & t \\ \text{subject to} & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- minimizing over some variables

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_i(x_1) \leq 0, \quad i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(x_1) \\ \text{subject to} & f_i(x_1) \leq 0, \quad i = 1, \ldots, m \end{array}$$

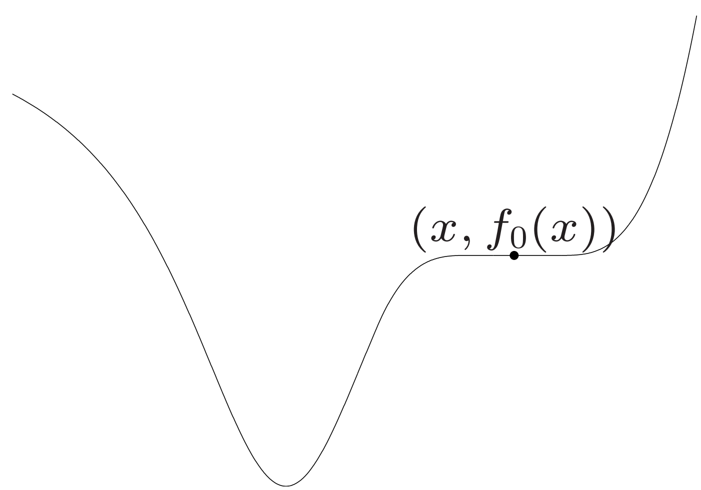
where $\tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$

# Quasiconvex optimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

with $f_0 : \mathbf{R}^n \to \mathbf{R}$ quasiconvex, $f_1, \ldots, f_m$ convex

can have locally optimal points that are not (globally) optimal



## convex representation of sublevel sets of $f_0$

if $f_0$ is quasiconvex, there exists a family of functions $\phi_t$ such that:

- $\phi_t(x)$ is convex in $x$ for fixed $t$
- $t$-sublevel set of $f_0$ is $0$-sublevel set of $\phi_t$, *i.e.*,

$$f_0(x) \leq t \quad \Longleftrightarrow \quad \phi_t(x) \leq 0$$

## example

$$f_0(x) = \frac{p(x)}{q(x)}$$

with $p$ convex, $q$ concave, and $p(x) \geq 0$, $q(x) > 0$ on $\mathbf{dom}\, f_0$

can take $\phi_t(x) = p(x) - tq(x)$:

- for $t \geq 0$, $\phi_t$ convex in $x$
- $p(x)/q(x) \leq t$ if and only if $\phi_t(x) \leq 0$

## quasiconvex optimization via convex feasibility problems

$$\phi_t(x) \leq 0, \qquad f_i(x) \leq 0, \quad i = 1, \ldots, m, \qquad Ax = b \tag{1}$$

- for fixed $t$, a convex feasibility problem in $x$
- if feasible, we can conclude that $t \geq p^\star$; if infeasible, $t \leq p^\star$

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*Bisection method for quasiconvex optimization*

**given** $l \leq p^\star$, $u \geq p^\star$, tolerance $\epsilon > 0$.
**repeat**
1. $t := (l + u)/2$.
2. Solve the convex feasibility problem (1).
3. **if** (1) is feasible, $u := t$; **else** $l := t$.

**until** $u - l \leq \epsilon$.

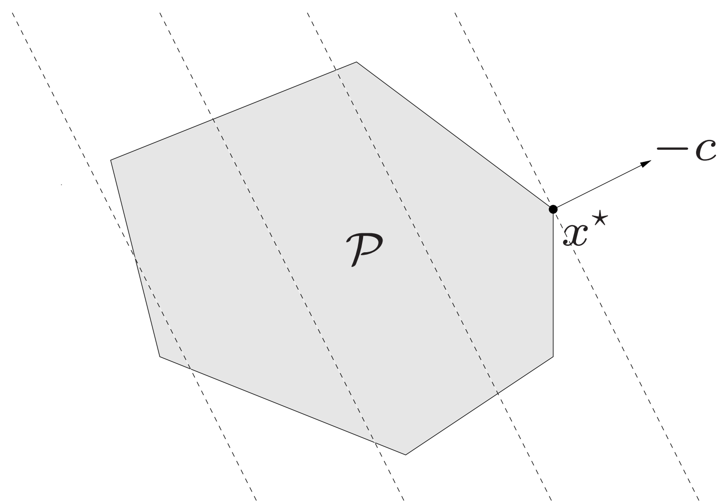
---

requires exactly $\lceil \log_2((u - l)/\epsilon) \rceil$ iterations (where $u$, $l$ are initial values)

# Linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



## Examples

diet problem: choose quantities $x_1, \ldots, x_n$ of $n$ foods

- one unit of food $j$ costs $c_j$, contains amount $a_{ij}$ of nutrient $i$
- healthy diet requires nutrient $i$ in quantity at least $b_i$

to find cheapest healthy diet,

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \succeq b, \quad x \succeq 0
\end{array}
$$

piecewise-linear minimization

$$
\text{minimize} \quad \max_{i=1,\ldots,m}(a_i^T x + b_i)
$$

equivalent to an LP

$$
\begin{array}{ll}
\text{minimize} & t \\
\text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \ldots, m
\end{array}
$$

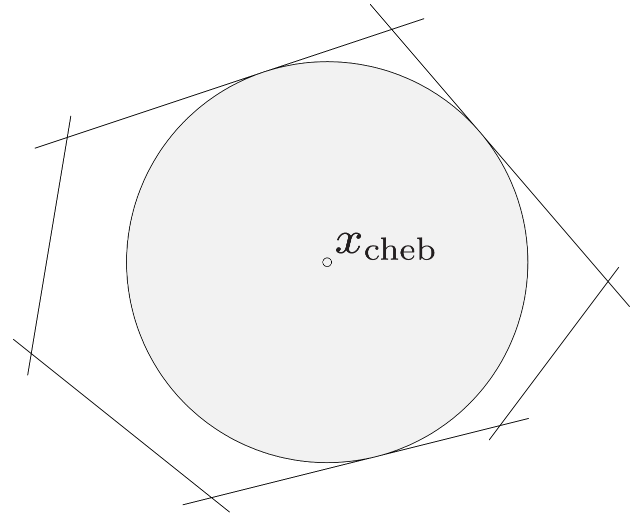
## Chebyshev center of a polyhedron

Chebyshev center of

$$\mathcal{P} = \{x \mid a_i^T x \leq b_i,\ i = 1, \ldots, m\}$$

is center of largest inscribed ball

$$\mathcal{B} = \{x_c + u \mid \|u\|_2 \leq r\}$$




- $a_i^T x \leq b_i$ for all $x \in \mathcal{B}$ if and only if

$$\sup\{a_i^T(x_c + u) \mid \|u\|_2 \leq r\} = a_i^T x_c + r\|a_i\|_2 \leq b_i$$

- hence, $x_c$, $r$ can be determined by solving the LP

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & a_i^T x_c + r\|a_i\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

# Linear-fractional program

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

linear-fractional program

$$
f_0(x) = \frac{c^T x + d}{e^T x + f}, \qquad \mathbf{dom}\, f_0(x) = \{x \mid e^T x + f > 0\}
$$

- a quasiconvex optimization problem; can be solved by bisection
- also equivalent to the LP (variables $y$, $z$)

$$
\begin{array}{ll}
\text{minimize} & c^T y + dz \\
\text{subject to} & Gy \preceq hz \\
& Ay = bz \\
& e^T y + fz = 1 \\
& z \geq 0
\end{array}
$$

## generalized linear-fractional program

$$f_0(x) = \max_{i=1,\ldots,r} \frac{c_i^T x + d_i}{e_i^T x + f_i}, \qquad \mathbf{dom}\, f_0(x) = \{x \mid e_i^T x + f_i > 0,\ i = 1, \ldots, r\}$$

a quasiconvex optimization problem; can be solved by bisection

**example**: Von Neumann model of a growing economy

$$\begin{array}{ll} \text{maximize (over } x,\, x^+\text{)} & \min_{i=1,\ldots,n} x_i^+/x_i \\ \text{subject to} & x^+ \succeq 0, \quad Bx^+ \preceq Ax \end{array}$$

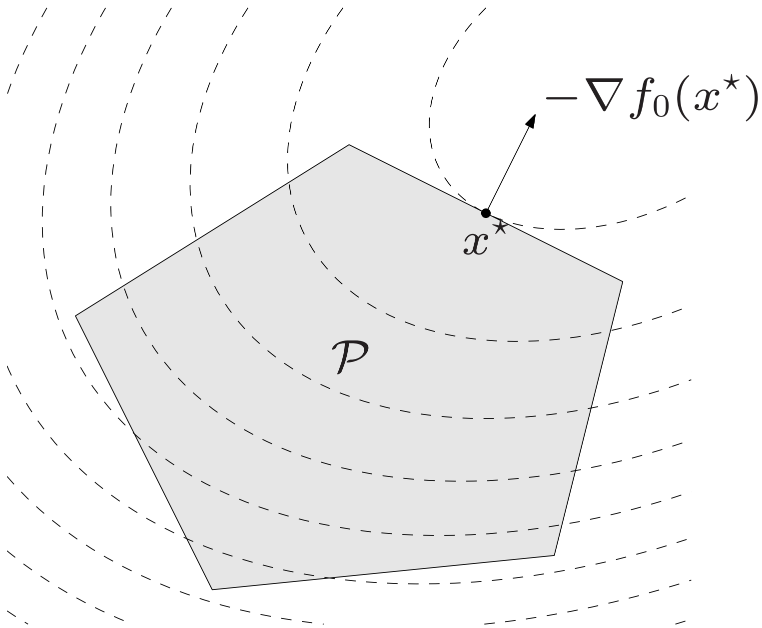
- $x, x^+ \in \mathbf{R}^n$: activity levels of $n$ sectors, in current and next period
- $(Ax)_i$, $(Bx^+)_i$: produced, resp. consumed, amounts of good $i$
- $x_i^+/x_i$: growth rate of sector $i$

allocate activity to maximize growth rate of slowest growing sector

# Quadratic program (QP)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- $P \in \mathbf{S}^n_+$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



# Examples

least-squares

$$\text{minimize} \quad \|Ax - b\|_2^2$$

- analytical solution $x^\star = A^\dagger b$ ($A^\dagger$ is pseudo-inverse)
- can add linear constraints, *e.g.*, $l \preceq x \preceq u$

linear program with random cost

$$\begin{array}{ll} \text{minimize} & \bar{c}^T x + \gamma x^T \Sigma x = \mathbf{E}\, c^T x + \gamma\, \mathbf{var}(c^T x) \\ \text{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

- $c$ is random vector with mean $\bar{c}$ and covariance $\Sigma$
- hence, $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$
- $\gamma > 0$ is risk aversion parameter; controls the trade-off between expected cost and variance (risk)

# Quadratically constrained quadratic program (QCQP)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- $P_i \in \mathbf{S}_+^n$; objective and constraints are convex quadratic
- if $P_1, \ldots, P_m \in \mathbf{S}_{++}^n$, feasible region is intersection of $m$ ellipsoids and an affine set

# Second-order cone programming

$$
\begin{array}{ll}
\text{minimize} & f^T x \\
\text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \\
& Fx = g
\end{array}
$$

($A_i \in \mathbf{R}^{n_i \times n}$, $F \in \mathbf{R}^{p \times n}$)

- inequalities are called second-order cone (SOC) constraints:

$$
(A_i x + b_i, c_i^T x + d_i) \in \text{second-order cone in } \mathbf{R}^{n_i+1}
$$

- for $n_i = 0$, reduces to an LP; if $c_i = 0$, reduces to a QCQP
- more general than QCQP and LP

# Robust linear programming

the parameters in optimization problems are often uncertain, *e.g.*, in an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i, \quad i = 1, \dots, m, \end{array}$$

there can be uncertainty in $c$, $a_i$, $b_i$

# Robust linear programming

the parameters in optimization problems are often uncertain, *e.g.*, in an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m, \end{array}$$

there can be uncertainty in $c$, $a_i$, $b_i$

two common approaches to handling uncertainty (in $a_i$, for simplicity)

- deterministic model: constraints must hold for all $a_i \in \mathcal{E}_i$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \text{ for all } a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m, \end{array}$$

- stochastic model: $a_i$ is random variable; constraints must hold with probability $\eta$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m \end{array}$$

## deterministic approach via SOCP

- choose an ellipsoid as $\mathcal{E}_i$:

$$\mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \le 1\} \qquad (\bar{a}_i \in \mathbf{R}^n, \quad P_i \in \mathbf{R}^{n\times n})$$

center is $\bar{a}_i$, semi-axes determined by singular values/vectors of $P_i$

- robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i \quad \forall a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m \end{array}$$

is equivalent to the SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \le b_i, \quad i = 1, \ldots, m \end{array}$$

(follows from $\sup_{\|u\|_2 \le 1} (\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2$)

## stochastic approach via SOCP

- assume $a_i$ is Gaussian with mean $\bar{a}_i$, covariance $\Sigma_i$ ($a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$)
- $a_i^T x$ is Gaussian r.v. with mean $\bar{a}_i^T x$, variance $x^T \Sigma_i x$; hence

$$\mathbf{prob}(a_i^T x \leq b_i) = \Phi\left(\frac{b_i - \bar{a}_i^T x}{\|\Sigma_i^{1/2} x\|_2}\right)$$

where $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2}\, dt$ is CDF of $\mathcal{N}(0, 1)$

- robust LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m, \end{array}$$

with $\eta \geq 1/2$, is equivalent to the SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \Phi^{-1}(\eta) \|\Sigma_i^{1/2} x\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

# Generalized inequality constraints

convex problem with generalized inequality constraints

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- $f_0 : \mathbf{R}^n \to \mathbf{R}$ convex; $f_i : \mathbf{R}^n \to \mathbf{R}^{k_i}$ $K_i$-convex w.r.t. proper cone $K_i$
- same properties as standard convex problem (convex feasible set, local optimum is global, etc.)

conic form problem: special case with affine objective and constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Fx + g \preceq_K 0 \\ & Ax = b \end{array}$$

extends linear programming ($K = \mathbf{R}^m_+$) to nonpolyhedral cones

# Semidefinite program (SDP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b \end{array}$$

with $F_i$, $G \in \mathbf{S}^k$

- inequality constraint is called linear matrix inequality (LMI)
- includes problems with multiple LMI constraints: for example,

$$x_1 \hat{F}_1 + \cdots + x_n \hat{F}_n + \hat{G} \preceq 0, \qquad x_1 \tilde{F}_1 + \cdots + x_n \tilde{F}_n + \tilde{G} \preceq 0$$

  is equivalent to single LMI

$$x_1 \left[ \begin{array}{cc} \hat{F}_1 & 0 \\ 0 & \tilde{F}_1 \end{array} \right] + x_2 \left[ \begin{array}{cc} \hat{F}_2 & 0 \\ 0 & \tilde{F}_2 \end{array} \right] + \cdots + x_n \left[ \begin{array}{cc} \hat{F}_n & 0 \\ 0 & \tilde{F}_n \end{array} \right] + \left[ \begin{array}{cc} \hat{G} & 0 \\ 0 & \tilde{G} \end{array} \right] \preceq 0$$

# LP and SOCP as SDP

## LP and equivalent SDP

LP: minimize $c^T x$
subject to $Ax \preceq b$

SDP: minimize $c^T x$
subject to $\mathbf{diag}(Ax - b) \preceq 0$

(note different interpretation of generalized inequality $\preceq$)

## SOCP and equivalent SDP

SOCP: minimize $f^T x$
subject to $\|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m$

SDP: minimize $f^T x$
subject to $\begin{bmatrix} (c_i^T x + d_i)I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, \ldots, m$

# Eigenvalue minimization

$$\text{minimize} \quad \lambda_{\max}(A(x))$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ (with given $A_i \in \mathbf{S}^k$)

equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A(x) \preceq tI \end{array}$$

- variables $x \in \mathbf{R}^n$, $t \in \mathbf{R}$
- follows from

$$\lambda_{\max}(A) \leq t \quad \Longleftrightarrow \quad A \preceq tI$$

# Matrix norm minimization

$$\text{minimize} \quad \|A(x)\|_2 = \left(\lambda_{\max}(A(x)^T A(x))\right)^{1/2}$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ (with given $A_i \in \mathbf{R}^{p \times q}$)

equivalent SDP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left[\begin{array}{cc} tI & A(x) \\ A(x)^T & tI \end{array}\right] \succeq 0 \end{array}$$

- variables $x \in \mathbf{R}^n$, $t \in \mathbf{R}$
- constraint follows from

$$\begin{aligned} \|A\|_2 \leq t \quad &\Longleftrightarrow \quad A^T A \preceq t^2 I, \quad t \geq 0 \\ &\Longleftrightarrow \quad \left[\begin{array}{cc} tI & A \\ A^T & tI \end{array}\right] \succeq 0 \end{aligned}$$

# Geometric programming

monomial function

$$f(x) = c x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++}$$

with $c > 0$; exponent $\alpha_i$ can be any real number

posynomial function: sum of monomials

$$f(x) = \sum_{k=1}^{K} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \cdots x_n^{\alpha_{nk}}, \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++}$$

geometric program (GP)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, \quad i = 1, \ldots, m \\ & h_i(x) = 1, \quad i = 1, \ldots, p \end{array}$$

with $f_i$ posynomial, $h_i$ monomial

# Geometric program in convex form

change variables to $y_i = \log x_i$, and take logarithm of cost, constraints

- monomial $f(x) = cx_1^{a_1} \cdots x_n^{a_n}$ transforms to

$$\log f(e^{y_1}, \ldots, e^{y_n}) = a^T y + b \qquad (b = \log c)$$

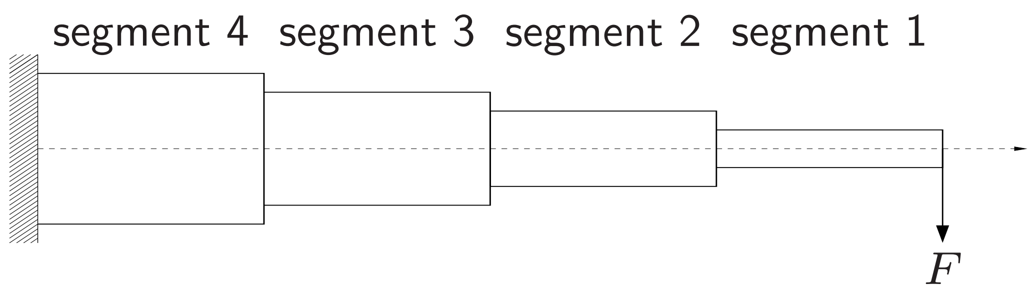
- posynomial $f(x) = \sum_{k=1}^K c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$ transforms to

$$\log f(e^{y_1}, \ldots, e^{y_n}) = \log \left( \sum_{k=1}^K e^{a_k^T y + b_k} \right) \qquad (b_k = \log c_k)$$

- geometric program transforms to convex problem

$$\begin{array}{ll} \text{minimize} & \log \left( \sum_{k=1}^K \exp(a_{0k}^T y + b_{0k}) \right) \\ \text{subject to} & \log \left( \sum_{k=1}^K \exp(a_{ik}^T y + b_{ik}) \right) \leq 0, \quad i = 1, \ldots, m \\ & Gy + d = 0 \end{array}$$

## Design of cantilever beam




- $N$ segments with unit lengths, rectangular cross-sections of size $w_i \times h_i$
- given vertical force $F$ applied at the right end

### design problem

| | |
|---|---|
| minimize | total weight |
| subject to | upper & lower bounds on $w_i$, $h_i$ |
| | upper bound & lower bounds on aspect ratios $h_i/w_i$ |
| | upper bound on stress in each segment |
| | upper bound on vertical deflection at the end of the beam |

variables: $w_i$, $h_i$ for $i = 1, \ldots, N$

## objective and constraint functions

- total weight $w_1 h_1 + \cdots + w_N h_N$ is posynomial
- aspect ratio $h_i/w_i$ and inverse aspect ratio $w_i/h_i$ are monomials
- maximum stress in segment $i$ is given by $6iF/(w_i h_i^2)$, a monomial
- the vertical deflection $y_i$ and slope $v_i$ of central axis at the right end of segment $i$ are defined recursively as

$$
\begin{aligned}
v_i &= 12(i - 1/2)\frac{F}{E w_i h_i^3} + v_{i+1} \\
y_i &= 6(i - 1/3)\frac{F}{E w_i h_i^3} + v_{i+1} + y_{i+1}
\end{aligned}
$$

  for $i = N, N-1, \ldots, 1$, with $v_{N+1} = y_{N+1} = 0$ ($E$ is Young's modulus)

  $v_i$ and $y_i$ are posynomial functions of $w$, $h$

## formulation as a GP

$$
\begin{array}{ll}
\text{minimize} & w_1 h_1 + \cdots + w_N h_N \\
\text{subject to} & w_{\max}^{-1} w_i \leq 1, \quad w_{\min} w_i^{-1} \leq 1, \quad i = 1, \ldots, N \\
& h_{\max}^{-1} h_i \leq 1, \quad h_{\min} h_i^{-1} \leq 1, \quad i = 1, \ldots, N \\
& S_{\max}^{-1} w_i^{-1} h_i \leq 1, \quad S_{\min} w_i h_i^{-1} \leq 1, \quad i = 1, \ldots, N \\
& 6iF\sigma_{\max}^{-1} w_i^{-1} h_i^{-2} \leq 1, \quad i = 1, \ldots, N \\
& y_{\max}^{-1} y_1 \leq 1
\end{array}
$$

note

- we write $w_{\min} \leq w_i \leq w_{\max}$ and $h_{\min} \leq h_i \leq h_{\max}$

$$
w_{\min}/w_i \leq 1, \qquad w_i/w_{\max} \leq 1, \qquad h_{\min}/h_i \leq 1, \qquad h_i/h_{\max} \leq 1
$$

- we write $S_{\min} \leq h_i/w_i \leq S_{\max}$ as

$$
S_{\min} w_i/h_i \leq 1, \qquad h_i/(w_i S_{\max}) \leq 1
$$

# Minimizing spectral radius of nonnegative matrix

## Perron-Frobenius eigenvalue $\lambda_{\mathrm{pf}}(A)$

- exists for (elementwise) positive $A \in \mathbf{R}^{n \times n}$
- a real, positive eigenvalue of $A$, equal to spectral radius $\max_i |\lambda_i(A)|$
- determines asymptotic growth (decay) rate of $A^k$: $A^k \sim \lambda_{\mathrm{pf}}^k$ as $k \to \infty$
- alternative characterization: $\lambda_{\mathrm{pf}}(A) = \inf\{\lambda \mid Av \preceq \lambda v \text{ for some } v \succ 0\}$

## minimizing spectral radius of matrix of posynomials

- minimize $\lambda_{\mathrm{pf}}(A(x))$, where the elements $A(x)_{ij}$ are posynomials of $x$
- equivalent geometric program:

$$
\begin{array}{ll}
\text{minimize} & \lambda \\
\text{subject to} & \sum_{j=1}^n A(x)_{ij} v_j / (\lambda v_i) \leq 1, \quad i = 1, \ldots, n
\end{array}
$$

variables $\lambda$, $v$, $x$

# Vector optimization

general vector optimization problem

$$\begin{array}{ll} \text{minimize (w.r.t. } K) & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) \leq 0, \quad i = 1, \ldots, p \end{array}$$

vector objective $f_0 : \mathbf{R}^n \to \mathbf{R}^q$, minimized w.r.t. proper cone $K \in \mathbf{R}^q$

convex vector optimization problem

$$\begin{array}{ll} \text{minimize (w.r.t. } K) & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

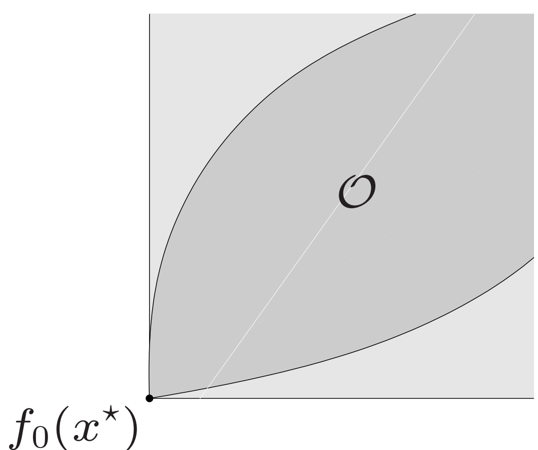
with $f_0$ $K$-convex, $f_1$, . . . , $f_m$ convex

# Optimal and Pareto optimal points

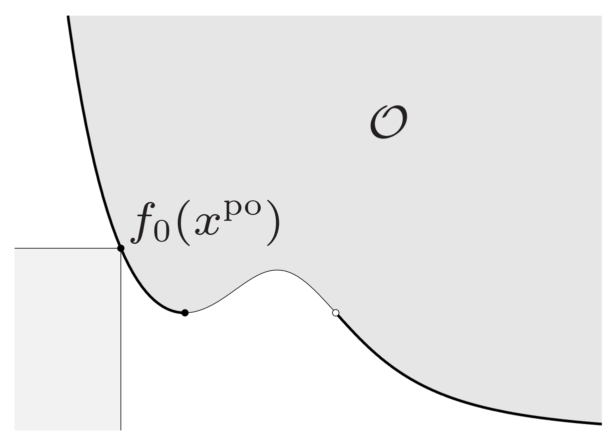
set of achievable objective values

$$\mathcal{O} = \{f_0(x) \mid x \text{ feasible}\}$$

- feasible $x$ is optimal if $f_0(x)$ is the minimum value of $\mathcal{O}$
- feasible $x$ is Pareto optimal if $f_0(x)$ is a minimal value of $\mathcal{O}$




$x^\star$ is optimal




$x^{\mathrm{po}}$ is Pareto optimal

# Multicriterion optimization

vector optimization problem with $K = \mathbf{R}^q_+$

$$f_0(x) = (F_1(x), \ldots, F_q(x))$$

- $q$ different objectives $F_i$; roughly speaking we want all $F_i$'s to be small
- feasible $x^\star$ is optimal if

$$y \text{ feasible} \quad \Longrightarrow \quad f_0(x^\star) \preceq f_0(y)$$

  if there exists an optimal point, the objectives are noncompeting

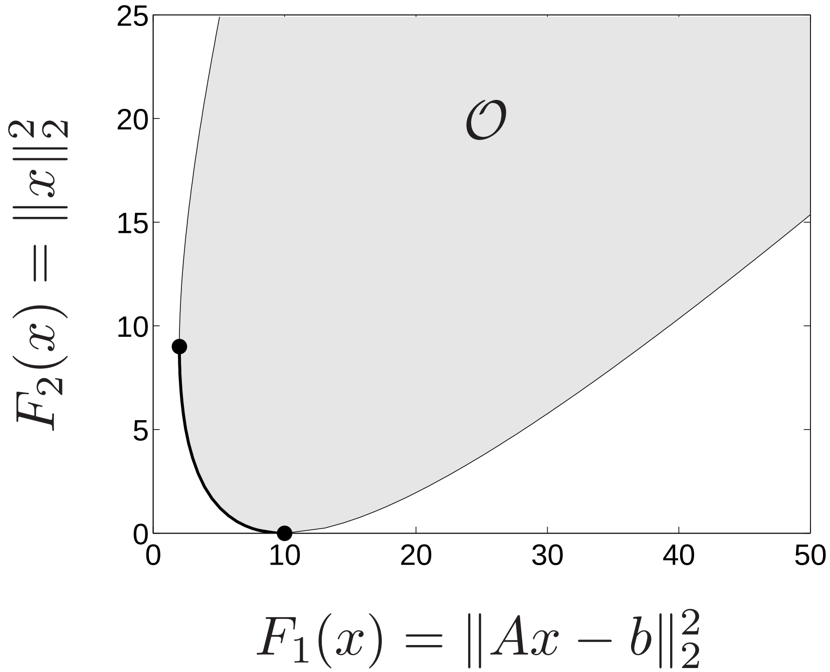
- feasible $x^{\text{po}}$ is Pareto optimal if

$$y \text{ feasible}, \quad f_0(y) \preceq f_0(x^{\text{po}}) \quad \Longrightarrow \quad f_0(x^{\text{po}}) = f_0(y)$$

  if there are multiple Pareto optimal values, there is a trade-off between the objectives

# Regularized least-squares

$$\text{minimize (w.r.t. } \mathbf{R}^2_+) \quad (\|Ax - b\|_2^2, \|x\|_2^2)$$



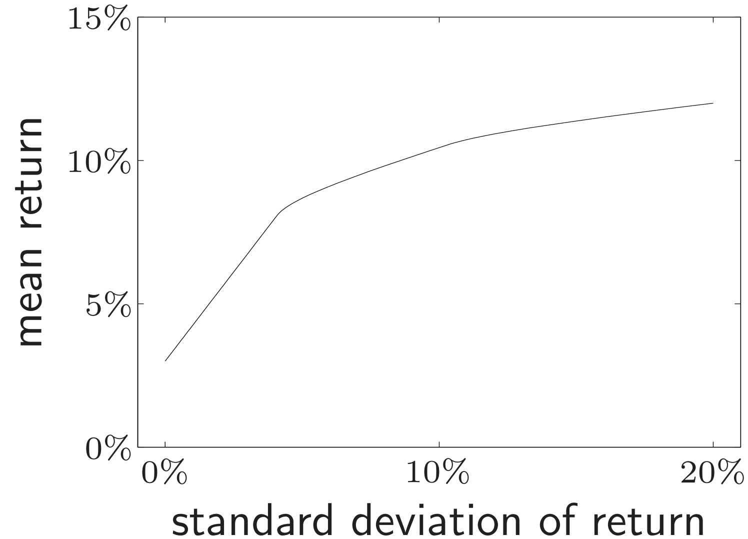

example for $A \in \mathbf{R}^{100\times 10}$; heavy line is formed by Pareto optimal points

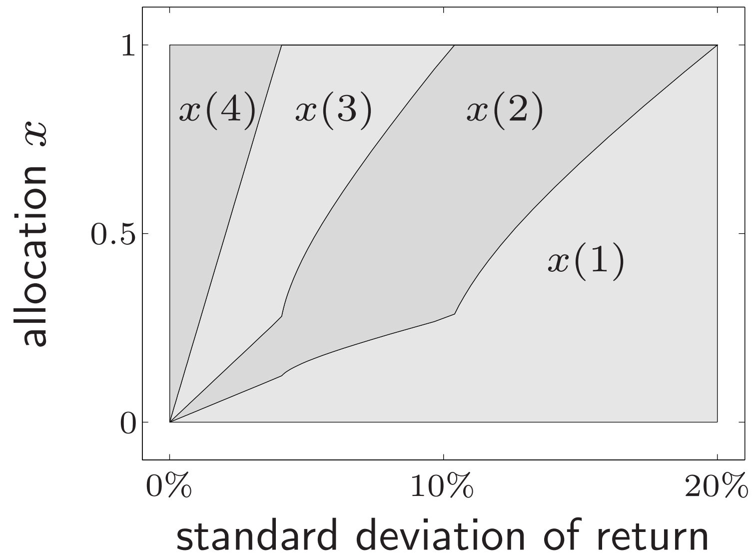
# Risk return trade-off in portfolio optimization

$$\begin{array}{ll} \text{minimize (w.r.t. } \mathbf{R}^2_+) & (-\bar{p}^T x, x^T \Sigma x) \\ \text{subject to} & \mathbf{1}^T x = 1, \quad x \succeq 0 \end{array}$$

- $x \in \mathbf{R}^n$ is investment portfolio; $x_i$ is fraction invested in asset $i$
- $p \in \mathbf{R}^n$ is vector of relative asset price changes; modeled as a random variable with mean $\bar{p}$, covariance $\Sigma$
- $\bar{p}^T x = \mathbf{E}\, r$ is expected return; $x^T \Sigma x = \mathbf{var}\, r$ is return variance

example



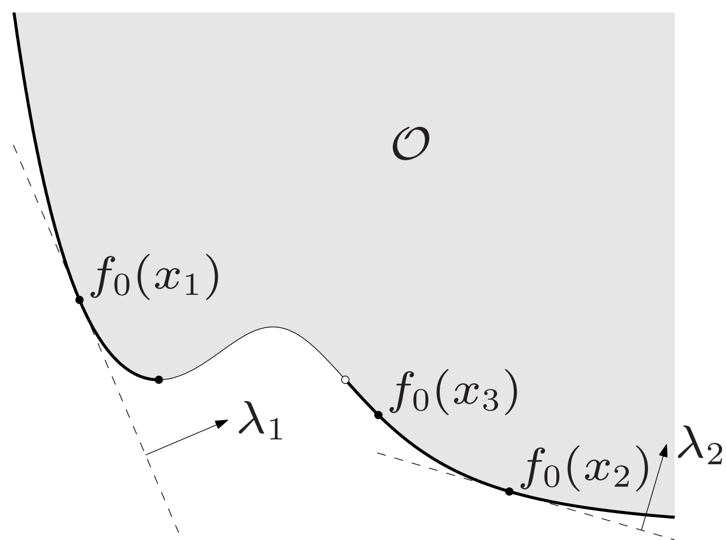



# Scalarization

to find Pareto optimal points: choose $\lambda \succ_{K^*} 0$ and solve scalar problem

$$\begin{array}{ll} \text{minimize} & \lambda^T f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

if $x$ is optimal for scalar problem, then it is Pareto-optimal for vector optimization problem




for convex vector optimization problems, can find (almost) all Pareto optimal points by varying $\lambda \succ_{K^*} 0$

# Scalarization for multicriterion problems

to find Pareto optimal points, minimize positive weighted sum

$$\lambda^T f_0(x) = \lambda_1 F_1(x) + \cdots + \lambda_q F_q(x)$$

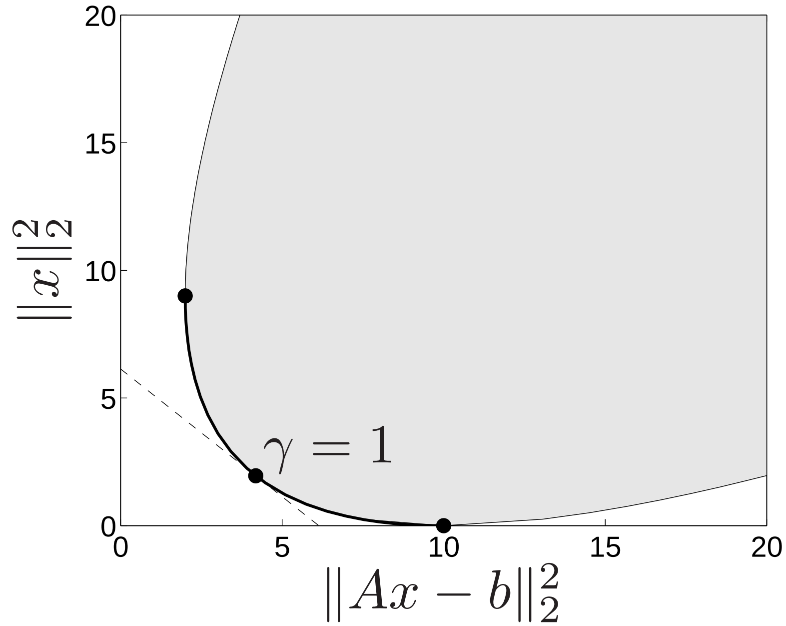
examples

- regularized least-squares problem of page 44

take $\lambda = (1, \gamma)$ with $\gamma > 0$

$$\text{minimize} \quad \|Ax - b\|_2^2 + \gamma \|x\|_2^2$$

for fixed $\gamma$, a LS problem



- risk-return trade-off of page 45

$$\begin{array}{ll} \text{minimize} & -\bar{p}^T x + \gamma x^T \Sigma x \\ \text{subject to} & \mathbf{1}^T x = 1, \quad x \succeq 0 \end{array}$$

for fixed $\gamma > 0$, a quadratic program