Equilibrium and Dynamics of Local Voltage Control in Distribution Systems

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***Abstract*— We consider a class of local volt/var control schemes where the control decision on the reactive power at a bus depends only on the local bus voltage. These local algorithms form a feedback dynamical system and collectively determine the bus voltages of a power network. We show that the dynamical system has a unique equilibrium by interpreting the dynamics as a distributed algorithm for solving a certain convex optimization problem whose unique optimal point is the system equilibrium. Moreover, the objective function serves as a Lyapunov function implying global asymptotic stability of the equilibrium. The optimization based model does not only provide a way to characterize the equilibrium, but also suggests a principled way to engineer the control. We apply the results to study the parameter setting for the inverter-based volt/var control in the proposed IEEE 1547.8 standard.**

I. Introduction

Traditionally voltages in a primary distribution system fluctuate slowly due to changes in demand which are relatively mild and predictable. Capacitor banks and under load tap changers (ULTC) are reconfigured a few times a day to stabilize voltages around their nominal values; see, e.g., [2], [3], [5]. The continued proliferation of distributed generation such as photovoltaic will introduce frequent, rapid, and random fluctuations in supply and voltages in a primary distribution system. Capacitor banks and ULTC alone may not be adequate to stabilize voltages in such an environment. Distributed energy resources such as photovoltaic systems are connected to the grid through inverters. Even though the current IEEE Standard 1547 requires unity power factor at the output of an inverter, the inverter hardware can easily optimize its reactive power output to help stabilize voltages. Indeed the IEEE Standards group is actively exploring a new inverter-based volt/var control. Unlike the capacity banks or ULTC, inverters can push and pull reactive power much faster, in a much finer granularity, and with low operation costs; see, e.g., [6], [8], [9]. They will enable realtime distributed volt/var control that is needed for the future power grid.

In this paper we study a class of inverter-based local volt/var control schemes that are motivated by the proposed 1547.8 standard [1]. These schemes set the reactive power at the output of an inverter based only on the local voltage deviation from its nominal value at a bus. Var control based on realtime voltage measurements has also been proposed in a transmission system; see, e.g., [7]. We use a linear branch flow model similar to the *Simplified DistFlow equations* introduced in [4]. The linear branch flow model and the local volt/var control form a closed loop dynamical system (Section II). We show that the dynamical system has a unique equilibrium point and characterize it as the unique optimal solution of a certain convex optimization problem (Section III). The optimization problem has a simple interpretation: the local volt/var control tries to achieve an optimal tradeoff between minimizing the cost of voltage deviations and minimizing the cost of reactive power provisioning. Moreover, the objective of the optimization problem serves as a Lyapunov function of the dynamical system under local volt/var control, implying global asymptotic stability of the equilibrium. We further provide a sufficient condition under which the dynamical system yields a contraction mapping, implying that it converges exponentially fast to the equilibrium. We apply these results to study the inverter-based volt/var control in IEEE 1547.8 standard [1], and discuss how to set the parameters for the proposed control functions (Section IV). The optimization based model does not only provide a way to characterize the equilibrium and establish the convergence of the local volt/var control, but also suggest a principled way to engineer the control. New design goals such as fairness and economic efficiency can be taken incorporated by engineering the global objective function in the optimization problem; and new control schemes with better dynamical properties can be designed based on various optimization algorithms, e.g., the gradient algorithms.

II. Network model and local voltage control

Consider a tree graph $G = \{\mathcal{N} \cup \{0\}, \mathcal{L}\}$ that represents a radial distribution network consisting of $n+1$ buses and a set $\mathcal{L}$ of lines between these buses. Bus 0 is the substation bus and is assumed to have a fixed voltage. Let $\mathcal{N} := \{1, \dots, n\}$. For each bus $i \in \mathcal{N}$, denote by $\mathcal{L}_i \subseteq \mathcal{L}$ the set of lines on the unique path from bus 0 to bus i, p_i^c and p_i^g the real power consumption and generation, and q_i^c and q_i^g the reactive power consumption and generation, respectively. Let v_i be the magnitude of the complex voltage (phasor) at bus i. For each line $(i, j) \in \mathcal{L}$, denote by r_{ij} and x_{ij} its resistance and reactance, and P_{ij} and Q_{ij} the real and reactive power from bus i to bus j respectively. Let ℓ_{ij} denote the squared magnitude of the complex branch current (phasor) from bus i to bus j. These notations are summarized in Table I. A quantity without subscript is usually a vector with appropriate components defined earlier, e.g., $v := (v_i, i \in \mathcal{N}), q^g := (q_i^g, i \in \mathcal{N})$.

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TABLE I
NOTATIONS

$\mathcal{N}$	Set of buses, excluding bus 0, labeled as {1, 2, ..., n}
$\mathcal{L}$	Set of all links representing the power lines
$\mathcal{L}_i$	Set of the links on the path form bus 0 to bus i
p_i^c, p_j^g	Real power consumption and generation at bus i
q_i^c, q_i^g	Reactive power consumption and generation at bus i
r_{ij}, x_{ij}	Resistance and reactance of line (i, j)
P_{ij}, Q_{ij}	Real and reactive power flows from i to j
v_i	Magnitude of complex voltage at bus i
ℓ_{ij}	Squared magnitude of complex current on (i, j)
$\beta(i) \subset \mathcal{N}$	Set of all descendants of node i

A. *Linearized branch flow model*

We adopt the following branch flow model introduced in [2], [3] (called *DistFlow equations* there) to model a radial distribution system:

$$\begin{aligned} P_{ij} &= p_j^c - p_j^g + \sum_{k:(j,k)\in\mathcal{L}} P_{jk} + r_{ij}\ell_{ij}, && (1a)\\ Q_{ij} &= q_j^c - q_j^g + \sum_{k:(j,k)\in\mathcal{L}} Q_{jk} + x_{ij}\ell_{ij}, && (1b)\\ v_j^2 &= v_i^2 - 2\left(r_{ij}P_{ij} + x_{ij}Q_{ij}\right) + \left(r_{ij}^2 + x_{ij}^2\right)\ell_{ij}, && (1c)\\ \ell_{ij}v_i &= P_{ij}^2 + Q_{ij}^2. && (1d) \end{aligned}$$

Following [4] we assume $\ell_{ij} = 0$ for all $(i, j) \in \mathcal{L}$ in (1). This approximation neglects the higher order real and reactive power loss terms. Since losses are typically much smaller than power flows P_{ij} and Q_{ij}, it only introduces a small relative error, typically on the order of 1%. We further assume that $v_i \approx 1$ so that we can set $v_j^2 - v_i^2 = 2(v_j - v_i)$ in equation (1c). This approximation introduces a small relative error of at most 0.25% (1%) if there is a 5% (10%) deviation in voltage magnitude.

With the above approximations the model (1) simplifies to the following linear model [4]:

$$\begin{aligned} P_{ij} &= \sum_{k\in\beta(j)} \left(p_k^c - p_k^g\right),\\ Q_{ij} &= \sum_{k\in\beta(j)} \left(q_j^c - q_j^g\right),\\ v_i - v_j &= r_{ij}P_{ij} + x_{ij}Q_{ij}, \end{aligned}$$

where $\beta(j)$ is the set of all descendants of node j including node j itself, i.e., $\beta(j) = \left\{i | \mathcal{L}_j \subseteq \mathcal{L}_i\right\}$. This yields an explicit solution for v_i in terms of v_0 (which is given and fixed):

$$\begin{aligned} v_0 - v_i &= \sum_{(j,k)\in\mathcal{L}_i} r_{jk}P_{jk} + \sum_{(j,k)\in\mathcal{L}_i} x_{jk}Q_{jk}\\ &= \sum_{(j,k)\in\mathcal{L}_i} r_{jk}\left(\sum_{h\in\beta(k)}\left(p_h^c - p_h^g\right)\right) + \sum_{(j,k)\in\mathcal{L}_i} x_{jk}\left(\sum_{h\in\beta(k)}\left(q_h^c - q_h^g\right)\right)\\ &= \sum_{j\in\mathcal{N}}\left(p_j^c - p_j^g\right)\left(\sum_{(h,k)\in\mathcal{L}_i\cap\mathcal{L}_j} r_{hk}\right) + \sum_{j\in\mathcal{N}}\left(q_j^c - q_j^g\right)\left(\sum_{(h,k)\in\mathcal{L}_i\cap\mathcal{L}_j} x_{hk}\right)\\ &= \sum_{j\in\mathcal{N}} R_{ij}\left(p_j^c - p_j^g\right) + \sum_{j\in\mathcal{N}} X_{ij}\left(q_j^c - q_j^g\right), \end{aligned}$$

where

$$\begin{aligned} R_{ij} &:= \sum_{(h,k)\in\mathcal{L}_i\cap\mathcal{L}_j} r_{hk},\\ X_{ij} &:= \sum_{(h,k)\in\mathcal{L}_i\cap\mathcal{L}_j} x_{hk}. \end{aligned} \qquad (2)$$

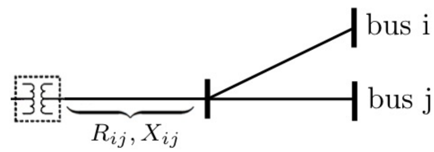


Fig. 1. $\mathcal{L}_i \cap \mathcal{L}_j$ for two arbitrary buses i, j in the network and the corresponding mutual voltage-to-power-injection sensitivity factors R_{ij}, X_{ij}.

Figure 1 gives an illustration of $\mathcal{L}_i \cap \mathcal{L}_j$ for two arbitrary buses i and j in a radial network and the corresponding R_{ij} and X_{ij}.[1] Define a resistance matrix $R = [R_{ij}]_{n\times n}$ and a reactance matrix $X = [X_{ij}]_{n\times n}$. Both matrices are symmetric. Using the matrices R and X the linearized branch flow model can be summarized compactly as

$$v = \overline{v}_0 + R(p^g - p^c) + X(q^g - q^c),$$

where $\overline{v}_0 = (v_0, \ldots, v_0)$ is an n-dimensional vector. In this paper we assume that $\overline{v}_0, p^c, p^g, q^c$ are given constants. The only variables are (column) vectors $v := (v_1, \ldots, v_n)$ of squared voltage magnitudes and $q^g := (q_1^g, \ldots, q_n^g)$ of reactive powers. Let $\tilde{v} = \overline{v}_0 + R(p^g - p^c) - Xq^c$, which is a constant vector. For notational simplicity in the rest of the paper we will ignore the superscript in q^g and write q instead. Then the linearized branch flow model reduces to the following simple form:

$$v = Xq + \tilde{v}. \qquad (3)$$

The following result is important for the rest of this paper.

Lemma 1: The matrices R and X are positive definite.

Proof: The proof will use the fact that the values of resistances and reactances of power lines in the network are all positive. Here we give a proof for the reactance matrix X, and exactly the same argument applies to the resistance matrix R.

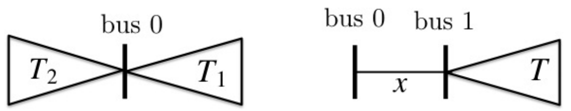
We prove by induction on the number k of buses in the network, excluding bus 0 (the root bus). The base case of $k = 1$ corresponds to a two-bus network with one line. Here X is obviously a positive scalar that is equal to the reactance of the line connecting the two buses.

Suppose that the theorem holds for all $k \le n$. For the case of $k = n+1$ we consider two possible network topologies as shown in Figure 2:

[1] Since

$$\begin{aligned} R_{ij} &= \frac{dv_i}{dp_j^g} = -\frac{dv_i}{dp_j^c},\\ X_{ij} &= \frac{dv_i}{dq_j^g} = -\frac{dv_i}{dq_j^c}, \end{aligned}$$

we refer to R_{ij}, X_{ij} as the mutual voltage-to-power-injection sensitivity factors.



(a) Case1: degree of bus 0 is greater than 1

(b) Case 2: degree of bus 0 is 1

Fig. 2. Two possible network structures

Case 1: bus 0 is of degree greater than 1. Split the network into two different trees rooted at bus 0, denoted by T_1 and T_2, each of which has no more than n buses excluding bus 0. Denote by Y and Z respectively the reactance matrices of T_1 and T_2. By induction assumption Y and Z are positive definite. Note that the set $\mathcal{L}_i$ of lines on the unique path from bus 0 to bus i must completely lie inside either T_1 or T_2, for all i. Therefore, by definition (2), the reactance matrix X of the network has the following block-diagonal form:

$$X_{ij} = \begin{cases} Y_{ij}, & i,j \in T_1 \\ Z_{ij}, & i,j \in T_2 \\ 0, & \text{otherwise} \end{cases} \quad \Rightarrow \quad X = \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix}.$$

Since Y and Z are positive definite, so is X.

Case2: bus 0 is of degree 1. Suppose without loss of generality that bus 0 is connected to bus 1. Denote by x the reactance of the line connecting buses 0 and 1, and T the tree rooted at bus 1. Tree T has $n-1$ buses excluding bus 1 (i.e., its root bus). Denote by Y the reactance matrix of T, and by induction assumption, Y is positive definite. Note that, for all nodes i in the network, the set $\mathcal{L}_i$ includes the single line that connects buses 0 and 1. Therefore, by definition (2), the reactance matrix X has the following form:

$$X_{ij} = \begin{cases} Y_{ij} + x, & i,j \in T^- \\ x, & \text{otherwise} \end{cases} \quad \Rightarrow \quad X = \begin{bmatrix} x & \dots x \\ \vdots & \vdots \\ x & \dots x \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Y \end{bmatrix},$$

where T^- denotes the set of nodes in tree T excluding the root bus 1. It is straightforward to verify that, when Y is positive definite and x is positive, X is positive definite. This concludes the proof. ∎

B. Local volt/var control

The goal of volt/var control on a distribution network is to provision reactive power injections $q := (q_1, \dots, q_n)$ in order to maintain the bus voltages $v := (v_1, \dots, v_n)$ to within a tight range around their nominal values v_i^{nom}, $i \in \mathcal{N}$. This can be modeled by a feedback dynamical system with state $(v(t), q(t))$ at discrete time t. A general volt/var control algorithm maps the current state $(v(t), q(t))$ to a new reactive power injections $q(t+1)$. The new $q(t+1)$ produces a new voltage magnitudes $v(t+1)$ according to (3). In this paper we focus on local volt/var control where each bus i makes an individual decision $q_i(t+1)$ based only on its own voltage $v_i(t)$.

Definition 1: A volt/var control function $f : \mathbb{R}^n \to \Omega$ is a collection of $f_i : \mathbb{R} \to \Omega_i$ functions that map the deviations $v_i(t) - v_i^{\text{nom}}$ of the current voltage from its nominal value to a new local control $q_i(t+1)$:

$$q_i(t+1) \;=\; f_i(v_i(t) - v_i^{\text{nom}}), \quad \forall i \in \mathcal{N}, \tag{4}$$

where $\Omega = \prod_{i=1}^{n} \Omega_i$, with $\Omega_i = \left\{ q_i \mid q_i^{\min} \le q_i \le q_i^{\max} \right\}$ the set of feasible reactive power injections at each bus $i \in \mathcal{N}$.

In summary, the dynamical system that models our volt/var control of a distribution network is given by:

$$v(t) \;=\; Xq(t) + \tilde{v}, \tag{5}$$

$$q(t+1) \;=\; f(v(t) - v^{\text{nom}}). \tag{6}$$

A fixed point of the above dynamical system represents an equilibrium operating point of the network.

Definition 2: (v^*, q^*) is called an equilibrium point, or a network equilibrium, if it is a fixed point of (5)–(6), i.e.,

$$v^* \;=\; Xq^* + \tilde{v},$$
$$q^* \;=\; f(v^* - v^{\text{nom}}).$$

III. Network equilibrium and dynamics

In this section we characterize the equilibrium point of local volt/var control (5)–(6) and study its dynamical properties.

A. Network equilibrium

The local volt/var control functions $f_i(v_i)$ are usually decreasing,[2] but are not always strictly monotone because of the deadband in control as well as the bounds on the available reactive power. We assume for each bus $i \in \mathcal{N}$ a symmetric deadband $[-\delta_i/2, \delta_i/2]$ around the origin with $\delta_i \ge 0$. Let $\overline{v}_i := \min\left\{v_i | f_i(v_i) = q_i^{\min}\right\}$, i.e., the lowest voltage deviation from the nominal value that draws reactive power $q_i^{\min}$, and let $\underline{v}_i := \max\left\{v_i | f_i(v_i) = q_i^{\max}\right\}$, i.e., the highest voltage deviation that draws reactive power $q_i^{\max}$. We make the following assumption:

A1: The local volt/var control functions f_i are nonincreasing over $\mathbb{R}$ and strictly decreasing and differentiable in $(\underline{v}_i, -\delta_i/2)$ and in $(\delta_i/2, \overline{v}_i)$.

Since f_i is nonincreasing a (generalized) inverse f_i^{-1} exists over $(q_i^{\min}, q_i^{\max})$. In particular, at the end points, we have

$$f^{-1}\left(q_i^{\min}\right) := \overline{v}_i \quad \text{and} \quad f^{-1}\left(q_i^{\max}\right) := \underline{v}_i$$

and, at the origin, we assign $f^{-1}(0) = 0$ in the deadband $[-\delta_i/2, +\delta_i/2]$. This may introduce a discontinuity at $q_i = 0$. See Figure 3 for an example f_i and Figure 4 for its inverse f_i^{-1}.

Define a cost function for each bus $i \in \mathcal{N}$:

$$C_i(q_i) \;:=\; -\int_0^{q_i} f_i^{-1}(q)\, dq.$$

This function is convex since f_i^{-1} is decreasing. Then, given any $v_i(t)$, $q_i(t+1)$ in (6) is the unique solution of a simple distributed optimization:

$$q_i(t+1) \;=\; \arg\min_{q_i^{\min} \le q_i \le q_i^{\max}} C_i(q_i) + q_i\left(v_i(t) - v_i^{\text{nom}}\right), \tag{7}$$

i.e., (6) and (7) are equivalent specification of $q_i(t+1)$.[3]

[2] Note that here we reload the notation, and use v_i to also denote the voltage magnitude deviation from the nominal value.

[3] They are equivalent specifications even if $v_i(t) - v_i^{\text{norm}}$ falls inside the deadband, i.e., if $|v_i(t) - v_i^{\text{norm}}| < \delta_i/2$. Under this situation, the set of subgradients of $C_i(q_i) + q_i(v_i(t) - v_i^{\text{nom}})$ at $q_i = 0$

$$\left[-\frac{\delta_i}{2} + \left(v_i(t) - v_i^{\text{norm}}\right), \frac{\delta_i}{2} + \left(v_i(t) - v_i^{\text{norm}}\right)\right]$$

Consider now the function $F : \Omega \to \mathbb{R}$:

$$F(q) \quad := \quad C(q) + \frac{1}{2} q^T X q + q^T \Delta\tilde{v}$$

where $C(q) = \sum_{i \in \mathcal{N}} C_i(q_i)$ and $\Delta\tilde{v} := \tilde{v} - v^{\text{norm}}$, and the global optimization problem:

$$\min_{q \in \Omega} \qquad F(q). \tag{8}$$

Theorem 1: Suppose A1 holds. Then there exists a unique equilibrium point. Moreover a point (v^*, q^*) is an equilibrium if and only if q^* is the unique optimal solution of (8) and $v^* = Xq^* + \tilde{v}$.

Proof: From Lemma 1 the matrix X is positive definite. This implies that the objective function $F(q)$ is strictly convex. Hence the first order optimality condition for (8) is both necessary and sufficient; moreover (8) has a unique optimal solution. We now relate it to the equilibrium point.

The gradient of F is the (column) vector

$$\begin{aligned} \nabla F(q) \quad &= \quad \nabla C(q) + Xq + \Delta\tilde{v} \\ &= \quad \nabla C(q) + (Xq + \tilde{v}) - v^{\text{norm}} \end{aligned}$$

where, from the definition of $C_i(q_i)$,

$$\nabla C(q) \quad = \quad \left[-f_1^{-1}(q_1) \quad \ldots \quad -f_n^{-1}(q_n) \right]^T .$$

Hence the first order optimality condition for (8) is:

$$q_i^* = f_i\left((Xq^* + \tilde{v}) - v^{\text{norm}} \right).$$

Hence a point (v^*, q^*) is an equilibrium if and only if q^* solves (8) and $v^* = Xq^* + \tilde{v}$. The existence and uniqueness of the optimal solution of (8) then implies that of the equilibrium (v^*, q^*). ■

With $v = Xq + \tilde{v}$, the objective can be written as $F(q, v) = C(q) + \frac{1}{2}(v - v^{nom})^T X^{-1} (v - v^{nom}) + \frac{1}{2} \Delta\tilde{v}^T X^{-1} \Delta\tilde{v}$. Note that the last term is a constant. Therefore the local volt/var control (5)-(6) tries to achieve an optimal tradeoff between minimizing the cost $\frac{1}{2}(v - v^{nom})^T X^{-1} (v - v^{nom})$ of voltage deviation and minimizing the cost $C(q)$ of reactive power provisioning.

B. Dynamics

We now study the dynamic properties of local volt/var control (5)–(6). We make the following assumption:

A2: The derivative of the control function f_i is bounded, i.e., there exists a finite α_i such that $|f_i'(v_i)| \leq \alpha_i$ for all v_i in the appropriate domain, for all $i \in \mathcal{N}$.

This assumption means that an infinitesimal change in voltage should not lead to a jump in reactive power.

Theorem 2: Suppose A1–A2 hold. if

$$\text{diag}\left(\frac{1}{\alpha_i} \right) \succ X, \tag{9}$$

contains 0, which is exactly the optimality condition at $q_i = 0$, and hence $q_i(t+1) = 0$. In the following we ignore such subdifferentiability issue with the understanding that subgradients should be used in place of gradients where functions are not differentiable.

i.e., if the matrix $\text{diag}\left(\alpha_i^{-1}\right) - X$ is positive definite, then local volt/var control (5)–(6) converges to the unique equilibrium point (v^*, q^*).

Proof: Recall that $C(q) = \sum_{i \in \mathcal{N}} C_i(q_i)$. Its Hessian

$$\nabla^2 C(q) = \text{diag}\left(-\frac{\partial f_i^{-1}(q_i)}{\partial q_i} \right).$$

By assumptions A1–A2 we have

$$\nabla^2 C(q) \geq \text{diag}\left(\frac{1}{\alpha_i} \right). \tag{10}$$

By the second order Taylor expansion,

$$\begin{aligned}
& F\left(q(t+1)\right) \\
= \;& C(q(t+1)) + \frac{1}{2} q^T(t+1) X q(t+1) + q^T(t+1)\Delta\tilde{v} \\
= \;& C(q(t)) + (\nabla C(q(t+1)))^T (q(t+1) - q(t)) \\
& - \frac{1}{2}(q(t+1) - q(t))^T \nabla^2 C(\tilde{q})(q(t+1) - q(t)) \\
& + \frac{1}{2} q^T(t) X q(t) + (q(t+1) - q(t))^T X q(t) \\
& + \frac{1}{2}(q(t+1) - q(t))^T X (q(t+1) - q(t)) \\
& + q^T(t)\Delta\tilde{v} - (q(t+1) - q(t))^T \Delta\tilde{v} \\
\leq \;& F(q(t)) \\
& + (\nabla C(q(t+1)) + Xq(t) + \Delta\tilde{v})^T (q(t+1) - q(t)) \\
& - \frac{1}{2}(q(t+1) - q(t))^T \left(\text{diag}\left(\alpha_i^{-1}\right) - X \right) (q(t+1) - q(t)) \\
\leq \;& F(q(t)) \\
& - \frac{1}{2}(q(t+1) - q(t))^T \left(\text{diag}\left(\alpha_i^{-1}\right) - X \right) (q(t+1) - q(t)),
\end{aligned} \tag{11}$$

where $\tilde{q} \in \{q \in \Omega \mid q = \theta q(t) + (1-\theta) q(t+1), 0 \leq \theta \leq 1\}$, the first inequality follows from (10), and the last inequality follows from (7).

Since $\text{diag}\left(\alpha_i^{-1}\right) \succ X$, the second term in (11) is strictly negative as long as $q(t+1) \neq q(t)$ and zero only if $q(t+1) = q(t)$. Since the fixed point to (5)–(6) is unique by Theorem 1, $q(t+1) = q(t)$ can only occur at the unique fixed point q^* (with $v^* = Xq^* + \tilde{v}$).

Thus we have shown the following:

- $F(q) \geq F(q^*)$ with equality if and only if $q = q^*$ by Theorem 1,
- $F(q(t+1)) \leq F(q(t))$ with equality if and only if $q(t+1) = q(t) = q^*$,

i.e., F is a discrete-time Lyapunov function for (5)–(6). Moreover one can extend the domain of each f_i^{-1} from $[q_i^{\min}, q_i^{\max}]$ to $\mathbb{R}$ in such a way that the above properties hold in the entire space and F is radially unbounded. The Lyapunov stability theorem then implies that q^* is globally asymptotically stable. ■

Since X is not only a positive definite matrix but also a positive matrix, $\text{diag}(\sum_{j \in \mathcal{N}} X_{ij}) \geq X$. This leads to a sufficient condition for the convergence of the local volt/var control.

Corollary 1: Suppose A1–A2 hold. If for all $i \in \mathcal{N}$

$$\alpha_i \sum_j X_{ij} \quad < \quad 1, \tag{12}$$

then local volt/var control (5)–(6) converges to the unique equilibrium point (v^*, q^*). Moreover, it converges exponentially fast to the equilibrium.

Proof: By condition (12), $\text{diag}\left(\alpha_i^{-1}\right) > \text{diag}\left(\sum_{j\in\mathcal{N}} X_{ij}\right)$. Since X is a positive definite as well as positive matrix, $\text{diag}\left(\sum_{j\in\mathcal{N}} X_{ij}\right) - X$ is diagonally dominant with non-negative diagonal entries, and is thus positive semidefinite. Thus $diag\left(\alpha_i^{-1}\right) > X$. By Theorem 2, local volt/var control (5)–(6) converges to the equilibrium point (v^*, q^*).

Now consider the equivalent system to (5)–(6):

$$q(t+1) \quad = \quad f(Xq(t) + \Delta\tilde{v}) \quad =: \quad g(q(t)).$$

We have

$$\frac{\partial g}{\partial q} \quad = \quad \text{diag}(f_i'(v_i))\, X$$

with $v_i := \sum_j X_{ij} q_j + \Delta\tilde{v}_i$. Condition (12) implies

$$\left\| \frac{\partial g}{\partial q} \right\|_\infty \quad < \quad 1$$

where the induced matrix norm $\|\cdot\|_\infty$ is the maximum row sum. Hence

$$\|g(q) - g(\hat{q})\|_\infty \quad \leq \quad \left\| \frac{\partial g}{\partial q} \right\|_\infty \cdot \|q - \hat{q}\|_\infty < \|q - \hat{q}\|_\infty,$$

i.e., g is a contraction. This implies that $(v(t), q(t))$ converges exponentially fast to the unique equilibrium point under (5)–(6). ■

Remark 1: We have reverse-engineered the local volt/var control (5)–(6), by showing that it is a distributed algorithm for solving a convex global optimization problem. The optimization based model (8) does not only provide a way to characterize the equilibrium and establish the convergence of the local volt/var control, but also suggests a principled way to engineer the control. New design goals such as fairness and economic efficiency can be taken incorporated by engineering the global objective function in (8); and new control schemes with better dynamical properties can be designed based on various optimization algorithms, e.g., the gradient algorithms.

IV. Case study: Inverter Control in IEEE 1547.8

We now apply the results of Section III to study the inverter-based volt/var control in IEEE 1547.8 standard [1] and discuss the parameter setting for the proposed control functions.

A. Reverse engineering 1547.8

The IEEE 1547 is currently being extended by the standards working group (IEEE 1547.8) to specify how to use inverters to assist in power quality control by adapting their reactive power generation. The methods being discussed in the latest draft [1] are:

1) Fixed power factor: the reactive power generation is directly proportional to the real power generation. This includes the traditional mode of operation with unity power factor where inverters are not allowed to inject or absorb reactive power under normal operating conditions.
2) Variable power factor: the reactive power generation depends not only on their active power output but also to X_{ii}/R_{ii} ratio at the point of connection.
3) Voltage-based reactive power control: an inverter monitors its terminal voltage and sets its reactive power generation based on a predefined volt-var curve.

In this paper we focus on the third approach. In particular, we study the piecewise linear volt/var curve shown in Figure 3 currently under discussion in the draft of the new IEEE 1547.8 [1]. This class of control functions are given by:

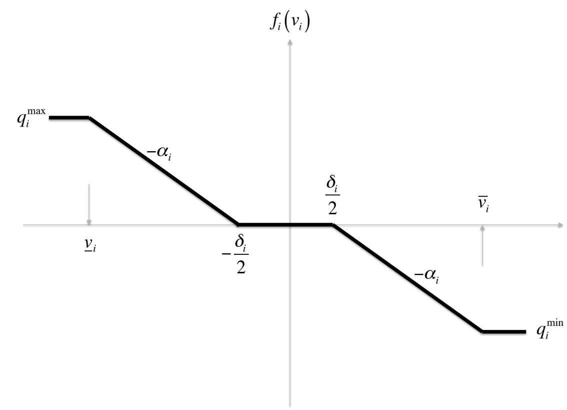


Fig. 3. Piecewise linear volt/var control curve discussed in IEEE 1547.8 [1].

$$f_i(v_i) := \left[-\alpha_i \left(v_i - \frac{\delta_i}{2} \right)^+ + \alpha_i \left(-v_i - \frac{\delta_i}{2} \right)^+ \right]_{q_i^{\min}}^{q_i^{\max}}, \qquad (13)$$

where $(x)^+ = \max\{x, 0\}$, and

$$[x]_a^b = \begin{cases} a & \text{for} \quad x \leq a, \\ x & \text{for} \quad a \leq x \leq b, \\ b & \text{for} \quad b \leq x. \end{cases}$$

They are specified by a deadband of width δ_i and two linear segment with a slope of $-\alpha_i$ for inverter i.

Following the procedure described in Section III-A, the inverse f_i^{-1} of the volt/var control curve over $[Q_i^{\min}, Q_i^{\max}]$ is illustrated in Figure 4 and given by:

$$f_i^{-1}(q_i) \quad := \quad \begin{cases} -\frac{q_i}{\alpha_i} + \frac{\delta_i}{2} & \text{for } q_i \in [q_i^{\min}, 0), \\ 0 & \text{for } q_i = 0, \\ -\frac{q_i}{\alpha_i} - \frac{\delta_i}{2} & \text{for } q_i \in (0, q_i^{\max}], \end{cases}$$

and the corresponding cost function $C_i(q_i) := \int_0^{q_i} f_i^{-1}(q)dq$ is shown in Figure 5 and is given by:

$$C_i(q_i) \quad = \quad \begin{cases} \frac{1}{2\alpha_i} q_i^2 - \frac{\delta_i}{2} q_i & \text{if} \quad q_i \in [q_i^{\min}, 0], \\ \frac{1}{2\alpha_i} q_i^2 + \frac{\delta_i}{2} q_i & \text{if} \quad q_i \in [0, q_i^{\max}]. \end{cases} \qquad (14)$$

B. Parameter setting

It has been suggested to set the slope of the piecewise linear control function to $\alpha_i = 1/X_{ii}$. This is a good choice if bus i is the only bus where the inverter-based volt/var control is employed. To see this, suppose that in the beginning $v_i = \tilde{v}_i > v_i^{nom} + \delta_i/2$. Then under the control (13), $q_i = -(v_i -$

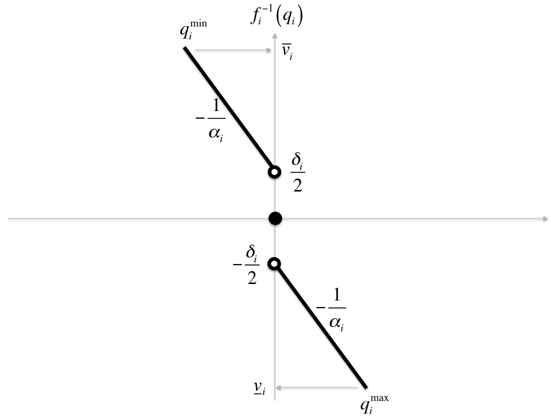


Fig. 4. The inverse f_i^{-1} of the volt/var control curve in Figure 3.

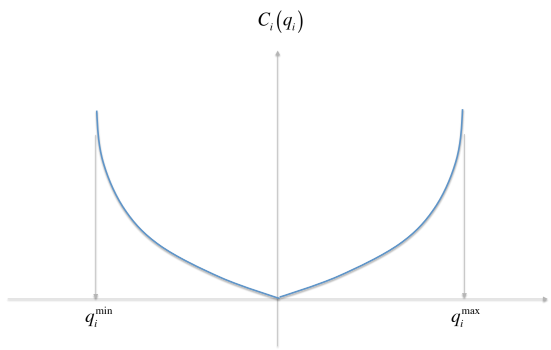


Fig. 5. The cost function $C_i(q_i)$ corresponding to f_i^{-1} of Figure 4.

$v_i^{nom} - \delta_i/2)/X_{ii}$ and $v_i = X_{ii}q_i + \tilde{v}_i = v_i^{nom} + \delta_i/2$. Therefore if the deadband δ_i is set to the range of the desired voltage, the volt/var control can bring the voltage of bus i to the desired range in just one step.

However the above choice does not take into consideration the impact of volt/var control at other buses. In particular, when $\alpha_i = 1/X_{ii}$, condition (9) in Theorem 2 does not hold, and the local control (13) may not converge to the equilibrium point. Instead, by Corollary 1, a convenient choice for α_i is to set $\alpha_i = (\sum_{j\in\mathcal{N}} X_{ij} + \epsilon)^{-1}$, where $\epsilon > 0$ can be used to control the convergence speed and a larger ϵ leads to faster response. Intuitively X_{ij} characterizes the sensitivity of bus j's voltage to reactive power injected at bus i, so if a bus has a larger impact to other buses (including itself), it should control its reactive power more cautiously, i.e., use a smaller α_i.

On the other hand, as mentioned in Section III-A, the local volt/var control tries to achieve an optimal tradeoff between minimizing the cost $\frac{1}{2}(v - v^{nom})^T X^{-1}(v - v^{nom})$ of voltage deviation and minimizing the cost $C(q)$ of reactive power provisioning. Seen from (14), a smaller α_i implies a steeper cost function of reactive power provisioning, which means that a larger voltage deviation may incur at the equilibrium. So a larger α_i and smaller ϵ is preferred for minimizing the voltage derivation. Therefore the ϵ value specifies the tradeoff between convergence speed and the voltage deviation. We will further study the optimal choice of ϵ and α_i in future work.

Seen from (14), a smaller deadband δ_i means a smaller marginal cost in reactive power provisioning and thus a smaller cost in reactive power provisioning. Intuitively, this implies that the smaller the deadband δ_i, the more bus i is willing/active to provision reactive power in order to achieve a narrower range of desired voltage.

The above discussion on parameter choice is based on the dynamical properties of the local volt/var control, as well as the impact of bus i's choice on itself; e.g., if the control at a bus has a smaller impact on itself or if a bus wants to achieve a narrower voltage range, it should be more active in reactive power provisioning. We can also set parameters to balance the contribution of a bus to the network versus its gain. For example, a larger X_{ii} means bus i can help more with regulating the voltages at other buses, so it may have a tighter range of desired voltage and set a smaller deadband. A fair choice for the deadband would be $\delta_i \propto 1/X_{ii}$.

V. Conclusion

We have studied a general class of local volt/var control schemes where the control decision on reactive power at a bus depends only on the voltage of that bus. By interpreting the resulting feedback dynamical system as a distributed algorithm for solving a convex global optimization problem, we have shown that the network has a unique equilibrium point. Moreover, the objective function serves as a Lyapunov function, leading to a simple condition that guarantees exponential convergence. The optimization based model also suggests a principled way to engineer the control. We have applied these results to the inverter-based volt/var control in the IEEE 1547.8 standard, and discuss how to set the parameters for the proposed control functions.

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