A New Perspective to Synchronization in Networks of Coupled Oscillators: Reverse Engineering and Convex Relaxation

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Abstract: We take a new approach to investigate synchronization in networks of coupled oscillators. We show that the coupled oscillator system when restricted to a proper region is a distributed partial primal-dual gradient algorithm for solving a well-defined convex optimization problem and its dual. We characterize conditions for synchronization solution of the KKT system of the optimization problem, based on which we derive conditions for synchronization equilibrium of the coupled oscillator network. This new approach reduces the hard problem of synchronization of coupled oscillators to a simple problem of verifying synchronization solution of a system of linear equations, and leads to a complete characterization of synchronization condition for the coupled oscillator network in an interesting and practically important region. Our synchronization condition is stated elegantly as the existence of solution for a system of linear equations, of which one best existing synchronization condition is a special sufficient condition case. In addition, we formulate a non-convex optimization problem with the force balance constraint for which the afore convex optimization problem is relaxation, and show that the coupled oscillator system is also a distributed algorithm for solving this non-convex problem. This has interesting implication on exact convex relaxation, and confirms the insight that a physical system usually solves a convex problem even though it may have a non-convex representation.

Keywords: Synchronization, reverse engineering, distributed algorithms, KKT system, convex relaxation, coupled oscillators, networks.

1. INTRODUCTION

The network of coupled oscillators and its synchronization is one of the most investigated network dynamical systems and behaviors. It has broad applications in various disciplines from biology and medicine to chemistry and physics and to engineering and social sciences; see, e.g., Wiesenfeld et al. (1998), Strogatz (2000), Varela et al. (2001), Winfree (2001), Kiss et al. (2002), Strogatz (2003), Jadbabaie et al. (2004), Boccaletti et al. (2006), Paley et al. (2007), Ha et al. (2010), Dorfler and Bullo (2011), Dorfler et al. (2013), Zhao et al. (2014), You and Chen (2014), Li et al. (2014), Mallada and Low (2014). Despite its broad applications, a complete or tight characterization of the condition for synchronization of coupled oscillators is mostly an open question.

In this paper, we consider a general coupled oscillator model that is partly motivated by the frequency dynamics and control in power networks: some of the oscillators are subject to the second-order Newtonian dynamics while the others are subject to the first-order kinematic dynamics, and they are sinusoidally coupled; see, e.g., You and Chen (2014) and Dorfler et al. (2013). This coupled oscillator model and its various special cases have been studied extensively; see, e.g., the above cited references and particularly Dorfler et al. (2013) for a brief review. In particular, Dorfler et al. (2013) presents an elegant closed-form condition for synchronization that significantly improves upon the existing conditions and is provably exact for various interesting network topologies and parameters.

Motivated by our prior work on the reverse engineering of the frequency control in the power network (You and Chen (2014)), we take a new approach to investigate synchronization of coupled oscillators. Specifically, we show that the coupled oscillator system when restricted to a proper region is a distributed partial primal-dual gradient algorithm for solving a well-defined convex optimization problem and its dual. We characterize conditions for synchronization solution of the KKT system of the optimization problem, based on which we derive conditions for synchronization equilibrium of the coupled oscillator network. This new approach reduces the hard problem of synchronization of coupled oscillators to a simple problem of verifying synchronization solution of a system of linear equations, and leads to a complete characterization of synchronization condition for the coupled oscillator network in an interesting and practically important region. Our synchronization condition is stated elegantly as the existence of solution for a system of linear equations, of which one synchronization condition of Dorfler et al. (2013) is a special sufficient condition case.

We then formulate a non-convex optimization problem with the force balance constraint for which the above-mentioned convex optimization problem is a relaxation. We show that the coupled oscillator system is also a distributed algorithm for solving this non-convex problem. This has an interesting implication on exact convex relaxation: a non-convex problem may be solved through solving its convex relaxation using a carefully chosen algorithm. This kind of exact convex relaxation is a bit different from the conventional one where the optimum of the convex problem is always a feasible point of the original non-convex problem, and confirms the insight that a physical system usually solves a convex problem even though it may have a non-convex representation.

2. SYSTEM MODEL

Consider a network modeled by a connected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, with a set $\mathcal{N}$ of nodes and a set $\mathcal{E}$ of undirected links connecting the nodes. Each node $i \in \mathcal{N}$ denotes an oscillator with phase $\theta_i \in \mathbb{R}$ and frequency $\omega_i = \dot{\theta}_i \in \mathbb{R}$, and each link $(i, j) \in \mathcal{E}$ (or $l \in \mathcal{E}$)[1] is associated with a weight or coupling constant $b_{ij} > 0$ (or $b_l > 0$). The node set is partitioned into two disjoint sets $\mathcal{N} = \mathcal{N}^1 \cup \mathcal{N}^2$. Consider the following coupled oscillator system:

$$M_i \dot{\omega}_i + F_i(\omega_i) = f_i - \sum_{\{j:(i,j)\in\mathcal{E}\}} b_{ij} \sin(\theta_i - \theta_j),\ i \in \mathcal{N}^1, \quad (1)$$

$$F_i(\omega_i) = f_i - \sum_{\{j:(i,j)\in\mathcal{E}\}} b_{ij} \sin(\theta_i - \theta_j),\ i \in \mathcal{N}^2, \quad (2)$$

where each oscillator $i \in \mathcal{N}^1$ follows the second-order Newtonian dynamics with an inertia constant $M_i > 0$ and each oscillator $i \in \mathcal{N}^2$ follows the first-order kinematic dynamics. Each oscillator $i \in \mathcal{N}$ is subject to a constant force of $f_i \in \mathbb{R}$ and a frequency-dependent damping of $F_i(\omega_i)$. The function $F_i(\cdot)$ is assumed to be Lipschitz continuous and strictly increasing.

The above coupled oscillator model (1)-(2) is partly motivated by the frequency dynamics and control in the power network, and a huge literature exists on the synchronization of this general system and its various special cases; see, e.g., Dorfler et al. (2013) and You and Chen (2014) and references therein. For instance, for the frequency dynamics of the power network, the set $\mathcal{N}^1$ is the set mechanical generators and $\mathcal{N}^2$ the set of load buses; f_i is the power inject or draw, $F_i(\omega_i) = D_i\omega_i$ with damping coefficient $D_i > 0$,[2] and M_i the generator inertia; and $b_{ij} = \frac{v_i v_j}{x_{ij}}$ with v_i the voltage magnitude at bus i and x_{ij} the reactance of power line (i, j); see, e.g., Bergen and Vittal (2000) and You and Chen (2014).

We aim to characterize conditions under which the network of coupled oscillators has a synchronization equilibrium and its stability.

[1] We use (i, j) and l interchangeably to denote a link in $\mathcal{E}$. Note that in this section (i, j) is an un-ordered pair, i.e., $(i, j) = (j, i)$. But from the next section on, $l \in \mathcal{E}$ is directed and $(i, j) \neq (j, i)$.

[2] Note that this damping term can result from frequency-sensitive load or frequency-based load or generation control. We can include more than one of such terms at each node as in You and Chen (2014), which will not change the structure of the problem and the results of this paper.

Definition 1. (*Synchronization equilibrium*) A synchronization equilibrium $(\omega, \boldsymbol{\theta} = \{\theta_i; i \in \mathcal{N}\}, \boldsymbol{\theta}^0 = \{\theta_i^0; i \in \mathcal{N}\})$ of the coupled oscillator system (1)-(2) is defined by the following relations:

$$\omega_i = \omega,\ i \in \mathcal{N}, \quad (3)$$

$$\theta_i(t) = \theta_i^0 + \omega t,\ i \in \mathcal{N}, \quad (4)$$

$$F_i(\omega) = f_i - \sum_{\{j:(i,j)\in\mathcal{E}\}} b_{ij} \sin(\theta_i - \theta_j),\ i \in \mathcal{N}, \quad (5)$$

where $\theta_i^0 \in [0, 2\pi),\ i \in \mathcal{N}$.

Motivated by the application in the power network where a security constraint $|\theta_i - \theta_j| < \frac{\pi}{2},\ (i, j) \in \mathcal{E}$ is usually imposed (see, e.g., Bergen and Hill (1981), Bergen and Vittal (2000), and Dorfler et al. (2014)), we are particularly interested in the synchronization equilibrium with $|\theta_i^0 - \theta_j^0| < \frac{\pi}{2},\ (i, j) \in \mathcal{E}$.

Definition 2. (*Phase Cohesiveness (Dorfler et al. (2013))*) Given $\gamma \in [0, \frac{\pi}{2})$, a synchronization equilibrium $(\omega, \boldsymbol{\theta}, \boldsymbol{\theta}^0)$ is γ phase cohesive if $|\theta_i^0 - \theta_j^0| \leq \gamma,\ (i, j) \in \mathcal{E}$.

2.1 Reverse Engineering of Network Dynamics with Linearized Coupling

Assume that the system is initially at a synchronization equilibrium with a "nominal" frequency ω^n and phases $\theta_i^n,\ i \in \mathcal{N}$ such that $|(\theta_i - \theta_j) - (\theta_i^n - \theta_j^n)| \ll 1,\ (i, j) \in \mathcal{E}$. Let $\tilde{b}_{ij} = b_{ij}\cos(\theta_i^n - \theta_j^n)$, and consider the following system with linearized coupling between oscillators:

$$M_i \dot{\omega}_i + F_i(\omega_i) = f_i - \sum_{\{j:(i,j)\in\mathcal{E}\}} p_{ij},\ i \in \mathcal{N}^1, \quad (6)$$

$$F_i(\omega_i) = f_i - \sum_{\{j:(i,j)\in\mathcal{E}\}} p_{ij},\ i \in \mathcal{N}^2, \quad (7)$$

$$\dot{p}_{ij} = \tilde{b}_{ij}(\omega_i - \omega_j),\ (i, j) \in \mathcal{E}. \quad (8)$$

In the power network application, $b_{ij}\sin(\theta_i - \theta_j)$ is the nonlinear power flow from bus i to bus j, and the above linearization corresponds to the assumption of small phase angle deviation; see, e.g., Bergen and Vittal (2000).

Let $d_i = F_i(\omega_i)$, and $F_i^{-1}(d_i)$ is well-defined because of F_i being strictly monotone. As in You and Chen (2014), we introduce a cost function corresponding to each damping term:

$$C_i(d_i) = \int F_i^{-1}(d_i)dd_i,\ i \in \mathcal{N}, \quad (9)$$

which is a strictly convex function by the assumption on the function F_i, and a convex optimization problem:

$$\min_{\mathbf{d},\mathbf{p}} \quad \sum_{i\in\mathcal{N}} C_i(d_i) \quad (10)$$

$$\text{subject to} \quad f_i = d_i + \sum_{\{j:(i,j)\in\mathcal{E}\}} p_{ij},\ i \in \mathcal{N}, \quad (11)$$

where $\mathbf{d} = \{d_i; i \in \mathcal{N}\}$ and $\mathbf{p} = \{p_{ij}; (i, j) \in \mathcal{E}\}$. The cost function $C_i(d_i)$ and problem (10)-(11) can have different interpretations, depending on specific applications. For instance, in the power network, $d_i = F_i(\omega_i)$ can be the primary frequency control and $C_i(d_i)$ is then the cost associated with the generation control, and problem (10)-(11) is a DC optimal power flow problem (You and Chen (2014)). Notice that there may be "operational" constraints on d_i. For instance, in the power network, there is a limited capacity for generation. These operational constraints can be incorporated implicitly through carefully defining the domain of function F_i or explicitly through adding to the optimization problem (10)-(11).

It has been shown that the system dynamics (6)-(8) can be seen as a distributed algorithm for solving the problem (10)-(11) and its dual; see, e.g., You and Chen (2014):

Theorem 3. (*Theorem 1 in You and Chen (2014) tailored to system* (6)-(8)) The set of saddle points of the Lagrangian of problem (10)-(11) is the set of synchronization equilibira of dynamical system (6)-(8). Moreover, the dynamics (6)-(8) is a partial primal-dual gradient algorithm for solving the problem (10)-(11) and its dual.

We have applied the above reverse engineering result to guide the design of new frequency control algorithms for the power system to not only recover nominal frequency but also achieve economic efficiency; see, e.g., You and Chen (2014) and Li et al. (2014). However, the above linearized model and reverse engineering result applies to the system with small phase angle derivation from an initial synchronization equilibrium, which is limited in applicability. An important question is if the above reverse engineering result can extend to the coupled oscillator system (1)-(2) with nonlinear coupling. In the next sections, we give a positive answer to this question, and use it to characterize the condition for synchronization in the network of coupled oscillators, as well as discuss its implication on convex relaxation.

3. REVERSE ENGINEERING AND SYNCHRONIZATION

We first introduce a few notations to simplify the presentation of the system and its analysis. Assigning an arbitrary direction to each link $l \in \mathcal{E}$, we define a $|\mathcal{N}| \times |\mathcal{E}|$ incidence matrix A with entry

$$A_{li} = \begin{cases} 1, & \text{if node } i \text{ is the source node of link } l \\ -1, & \text{if node } i \text{ is the sink node of link } l\ . \\ 0, & \text{otherwise} \end{cases}$$

Since $\mathcal{G}$ is connected, we have $\text{Rank}(A) = |\mathcal{N}| - 1$ and $\text{Ker}(A^T) = \text{span}(\mathbf{1}_{|\mathcal{N}|})$; see, e.g., Biggs (1993). With the incident matrix, we can rewrite the problem (10)-(11) as:

$$\min_{\mathbf{d},\mathbf{p}} \quad \sum_{i\in\mathcal{N}} C_i(d_i) \tag{12}$$

$$\text{subject to} \quad \mathbf{f} = \mathbf{d} + A\mathbf{p}, \tag{13}$$

where $\mathbf{f} = \{f_i; i \in \mathcal{N}\}$. Notice that the coupling $b_{ij}\sin(\theta_i - \theta_j)$ between nodes i and j is bounded by $\pm b_{ij}$. This implies that an additional constraint should be imposed on the problem (12)-(13):

$$-\mathbf{b} \preceq \mathbf{p} \preceq \mathbf{b}, \tag{14}$$

where $\mathbf{b} = \{b_l; l \in \mathcal{E}\}$. The problem (12)-(14) is convex, so all its optima are global optima. We further assume that the problem (12)-(14) is strictly feasible.

Lemma 4. The problem (12)-(14) may have multiple (global) optima in $\mathbf{p}$, but has an unique optimum in $\mathbf{d}$.

Proof. The objective function is not strictly convex in all the decision variable, so the problem (12)-(14) may have multiple global optima. Suppose that $\mathbf{d}$ can take two values $\bar{\mathbf{d}}$ and $\hat{\mathbf{d}}$ at optima. Since the objective function is strictly convex in $\mathbf{d}$, we have $\sum_{i\in\mathcal{N}} C_i(d_i) < \alpha \sum_{i\in\mathcal{N}} C_i(\bar{d}_i) + (1-\alpha)\sum_{i\in\mathcal{N}} C_i(\hat{d}_i)$ for any $\mathbf{d} = \alpha\bar{\mathbf{d}} + (1-\alpha)\hat{\mathbf{d}},\ 0 < \alpha < 1$. This contradicts the fact that $\bar{\mathbf{d}}$ and $\hat{\mathbf{d}}$ are optima. So, the problem (12)-(14) has an unique optimum in $\mathbf{d}$.

3.1 Synchronization Solution of the KKT System

Introduce Lagrangian multiplier λ_i for each constraint in (13), and write down the KKT condition of the problem (12)-(14) (see, e.g., Boyd and Vandenberghe (2004)):

$$f_i = d_i + \sum_{l\in\mathcal{E}} A_{il} p_l,\ i \in \mathcal{N}, \tag{15}$$

$$d_i = F_i(\lambda_i),\ i \in \mathcal{N}, \tag{16}$$

$$p_l = b_l,\ \text{if } \lambda_{s_l} > \lambda_{d_l},\ l \in \mathcal{E}, \tag{17}$$

$$p_l = -b_l,\ \text{if } \lambda_{s_l} < \lambda_{d_l},\ l \in \mathcal{E}, \tag{18}$$

$$p_l \in [-b_l, b_l],\ \text{if } \lambda_{s_l} = \lambda_{d_l},\ l \in \mathcal{E}, \tag{19}$$

where s_l and d_l denote the source and sink nodes of link l respectively. For the reason that will become clear later, we focus on those "synchronization" solutions to the above KKT system.

Definition 5. A solution to the KKT system (15)-(19) is said to be a synchronization solution if $\lambda_i = \lambda$ for all $i \in \mathcal{N}$.

It is obvious that, if there is a synchronization solution, λ is uniquely determined by $\sum_{i\in\mathcal{N}} f_i = \sum_{i\in\mathcal{N}} F_i(\lambda)$. Let $\bar{f}_i = f_i - F_i(\lambda)$. Then $\sum_{i\in\mathcal{N}} \bar{f}_i = 0$, and at a synchronization solution

$$\bar{\mathbf{f}} = A\mathbf{p}, \tag{20}$$

$$-\mathbf{b} \preceq \mathbf{p} \preceq \mathbf{b}, \tag{21}$$

where $\bar{\mathbf{f}} = \{\bar{f}_i; i \in \mathcal{N}\}$. From equation (20) we have

$$\mathbf{p} = A^T(AA^T)^\dagger \bar{\mathbf{f}} + \bar{\mathbf{p}}, \tag{22}$$

$$\bar{\mathbf{p}} \in \text{Ker}(A), \tag{23}$$

where '$\dagger$' denotes Moore-Penrose pseudo inverse, and AA^T and its pseudo inverse satisfies $AA^T(AA^T)^\dagger = (AA^T)^\dagger AA^T = I_{|\mathcal{N}|} - \frac{1}{|\mathcal{N}|}\mathbf{1}_{|\mathcal{N}|\times|\mathcal{N}|}$. The space $\text{Ker}(A)$ is related to the cycles in the network; see, e.g., Biggs (1993).

Theorem 6. The following three statements are equivalent:

(1) There exits at least one $\bar{\mathbf{p}} \in \text{Ker}(A)$ such that $-\mathbf{b} \preceq A^T(AA^T)^\dagger\bar{\mathbf{f}} + \bar{\mathbf{p}} \preceq \mathbf{b}$.
(2) The KKT system (15)-(19) has a synchronization solution.
(3) All the solutions of the KKT system (15)-(19) are synchronization solutions.

Proof. The equivalence of statements (1) and (2) is already shown in the above. The statement (3) obviously implies statement (2). Now, suppose that (2) holds but (3) does not, i.e., there exists another solution that is not a synchronization solution. Thus, the problem (12)-(14) has two different solutions in terms of $\mathbf{d}$ by equation (16), which contradicts Lemma 4. So, (2) implies (3), and thus statements (2) and (3) are equivalent.

The statement (1) of Theorem 6 gives a sufficient and necessary condition for the synchronization solution of the KKT system (15)-(19). To verify this condition is a linear programming (LP) problem, for which efficient algorithms exit; see, e.g., Boyd and Vandenberghe (2004).

Corollary 7. A sufficient condition for the existence of the synchronization solution of the KKT system (15)-(19) is given by

$$||(\text{diag}(\mathrm{b}))^{-1}A^T(AA^T)^\dagger\bar{\mathbf{f}}||_\infty \le 1, \tag{24}$$

where '$||\cdot||_\infty$' denotes infinity norm.

Proof. Let $\bar{\mathbf{p}} = 0$. If condition (24) holds, then $-\mathbf{b} \preceq \mathbf{p} = A^T(AA^T)^\dagger\bar{\mathbf{f}} \preceq \mathbf{b}$. So, $\mathbf{p}$ is a synchronization solution to the KKT system (15)-(19).

The condition (24) is easier to verify. It is also necessary for synchronization solution in certain networks with special structure, e.g., for the tree network where $\text{Ker}(A) = \{\mathbf{0}\}$. An interesting question is how tight the condition (24) is for general networks, compared with the sufficient and necessary condition in the statement (1) of Theorem 6. We will investigate related issues in future work.

3.2 Primal-Dual Gradient Algorithm

Let $\boldsymbol{\lambda} = \{\lambda_i; i \in \mathcal{N}\}$, and consider the Lagrangian for the problem (12)-(14):

$$L(\mathbf{d},\mathbf{p};\boldsymbol{\lambda}) = \sum_{i\in\mathcal{N}} C_i(d_i) + \boldsymbol{\lambda}^T(\mathbf{f} - \mathbf{d} - A\mathbf{p}). \tag{25}$$

A saddle point of L is a primal-dual optimum of the problem (12)-(14) and its dual (see, e.g., Boyd and Vandenberghe (2004)), and moreover, the saddle point is unique in $\boldsymbol{\lambda}$ by Lemma 4 and the strict monotonicity of the functions F_i.

Define a reduced Lagrangian:

$$\bar{L}(\mathbf{p};\boldsymbol{\lambda}^1) = \max_{\boldsymbol{\lambda}^2}\min_{\mathbf{d}} L(\mathbf{d},\mathbf{p};\boldsymbol{\lambda}), \tag{26}$$

where $\boldsymbol{\lambda}^1 = \{\lambda_i; i \in \mathcal{N}^1\}$ and $\boldsymbol{\lambda}^2 = \{\lambda_i; i \in \mathcal{N}^2\}$. From the inner minimization in (26) we have

$$d_i = F_i(\lambda_i),\ i \in \mathcal{N}. \tag{27}$$

The function $\min_{\mathbf{d}} L(\mathbf{d},\mathbf{p};\boldsymbol{\lambda})$ is strictly concave and continuously differentiable in $\boldsymbol{\lambda}$ by the assumption on the functions F_i. From the outer maximization we have

$$F_i(\lambda_i) = f_i - \sum_{l\in\mathcal{E}} A_{il}p_l,\ i \in \mathcal{N}^2. \tag{28}$$

Since $\min_{\mathbf{d}} L(\mathbf{d},\mathbf{p};\boldsymbol{\lambda})$ is strictly concave in $\boldsymbol{\lambda}$, the reduced Lagrangian $\bar{L}$ is strictly concave in $\boldsymbol{\lambda}^1$.

Applying the continuous-time primal-dual gradient algorithm (aka, saddle point dynamics) to the reduced Lagrangian, we have

$$\dot{p}_l = -\epsilon_l\frac{\partial\bar{L}}{\partial p_l} = \sqrt{b_l^2 - p_l^2}\,(\lambda_{s_l} - \lambda_{d_l}), l \in \mathcal{E}, \tag{29}$$

$$\dot{\lambda}_i = \gamma_i\frac{\partial\bar{L}}{\partial\lambda_i} = \frac{1}{M_i}(f_i - F_i(\lambda_i) - \sum_{l\in\mathcal{E}} A_{il}p_l), i \in \mathcal{N}^1, \tag{30}$$

where we have chosen specific scaling factors $\epsilon_l = \sqrt{b_l^2 - p_l^2}$ and $\gamma_i = \frac{1}{M_i}$. Notice that in equation (29) the choice of the scaling factor ensures that the constraint (14) is satisfied. As $dp_l/\sqrt{b_l^2 - p_l^2} = d\arcsin(p_l/b_l)$, if we identify λ_i with ω_i, the algorithm (27)-(30) is equivalent to the dynamical system (1)-(2) with the phases being restricted to $|\theta_i - \theta_j| < \pi/2,\ (i,j) \in \mathcal{E}$. We thus have the following result.

Theorem 8. If identifying λ_i with ω_i for all $i \in \mathcal{N}$, the network dynamics (1)-(2) in the region defined by $|\theta_i - \theta_j| < \pi/2,\ (i,j) \in \mathcal{E}$ is a distributed partial primal-dual gradient algorithm for solving the following problem and its dual:

$$\min_{\mathbf{d},\mathbf{p}} \quad \sum_{i\in\mathcal{N}} C_i(d_i) \tag{31}$$

$$\text{subject to} \quad \mathbf{f} = \mathbf{d} + A\mathbf{p}, \tag{32}$$

$$-\mathbf{b} \prec \mathbf{p} \prec \mathbf{b}. \tag{33}$$

Moreover, the set of synchronization equilibria of the dynamical system (1)-(2) in the region defined by $|\theta_i - \theta_j| < \pi/2,\ (i,j) \in \mathcal{E}$ is a subset of the set of saddle points of the Lagrangian L.

We will study the synchronization equilibrium and its stability of the network of coupled oscillators (1)-(2) from *the perspective that it is a primal-dual gradient algorithm for solving the problem* (31)-(33) *and its dual*, i.e., we will study the network dynamics (1)-(2) through studying the algorithm (27)-(30). For this purpose, in the rest of this paper we will assume that there exits at least one $\bar{\mathbf{p}} \in \text{Ker}(A)$ such that $-\mathbf{b} \prec A^T(AA^T)^\dagger\bar{\mathbf{f}} + \bar{\mathbf{p}} \prec \mathbf{b}$, under which all the primal-dual optima of the problem (31)-(33) and its dual are synchronization solutions by Theorem 6. We will also use ω_i and λ_i interchangeably from now on.

3.3 Synchronization Equilibrium and Its Stability

We first study the convergence of the primal-dual gradient algorithm (27)-(30) in the region defined by $-\mathbf{b} \prec \mathbf{p} \prec \mathbf{b}$.

Theorem 9. The primal-dual gradient algorithm (27)-(30) converges *locally* to a primal-dual optimum of the problem (31)-(33) and its dual.

Proof. Let $(\mathbf{p}^*, \boldsymbol{\lambda}^*)$ be a primal-dual optimum [3] of the problem (31)-(33) and its dual. Consider the Lyapunov function:

$$\begin{aligned}&U(\mathbf{p},\boldsymbol{\lambda}^1;\mathbf{p}^*,\boldsymbol{\lambda}^{1*})\\ &=\sum_{l\in\mathcal{E}}\int_{p_l^*}^{p_l}\frac{q_l - p_l^*}{\sqrt{b_l^2 - q_l^2}}dq_l + \sum_{i\in\mathcal{N}^1}\frac{M_i}{2}(\lambda_i - \lambda_i^*)^2,\end{aligned} \tag{34}$$

which is strictly convex if $-\mathbf{b} \prec \mathbf{p} \prec \mathbf{b}$. Consider its Lie-derivative under the algorithm (27)-(30):

$$\begin{aligned}&\dot{U}(\mathbf{p},\boldsymbol{\lambda}^1;\mathbf{p}^*,\boldsymbol{\lambda}^{1*})\\ &= -(\mathbf{p} - \mathbf{p}^*)^T\nabla_{\mathbf{p}}\bar{L} + (\boldsymbol{\lambda}^1 - \boldsymbol{\lambda}^{1*})^T\nabla_{\boldsymbol{\lambda}^1}\bar{L}\\ &\le \bar{L}(\mathbf{p}^*;\boldsymbol{\lambda}^1) - \bar{L}(\mathbf{p};\boldsymbol{\lambda}^1) + \bar{L}(\mathbf{p};\boldsymbol{\lambda}^1) - \bar{L}(\mathbf{p};\boldsymbol{\lambda}^{1*}) \qquad (35)\\ &= \bar{L}(\mathbf{p}^*;\boldsymbol{\lambda}^1) - \bar{L}(\mathbf{p};\boldsymbol{\lambda}^{1*})\\ &= \bar{L}(\mathbf{p}^*;\boldsymbol{\lambda}^1) - \bar{L}(\mathbf{p}^*;\boldsymbol{\lambda}^{1*}) + \bar{L}(\mathbf{p}^*;\boldsymbol{\lambda}^{1*}) - \bar{L}(\mathbf{p};\boldsymbol{\lambda}^{1*})\\ &\le 0,\end{aligned} \tag{36}$$

[3] Notice that $\boldsymbol{\lambda}^* = \lambda\mathbf{1}_{|\mathcal{N}|}$.

where inequality (35) follows from the fact that $\bar{L}$ is convex in $\mathbf{p}$ and concave in $\boldsymbol{\lambda}^1$, and inequality (36) from the fact that $(\mathbf{p}^*;\boldsymbol{\lambda}^{1*})$ is a saddle point of $\bar{L}$. Notice that if $\dot{U}(\mathbf{p},\boldsymbol{\lambda}^1;\mathbf{p}^*,\boldsymbol{\lambda}^{1*}) = 0$, then all the inequalities become equality, and $\bar{L}(\mathbf{p}^*;\boldsymbol{\lambda}^1) = \bar{L}(\mathbf{p}^*;\boldsymbol{\lambda}^{1*})$ and $\bar{L}(\mathbf{p}^*;\boldsymbol{\lambda}^{1*}) = \bar{L}(\mathbf{p};\boldsymbol{\lambda}^{1*})$. From LaSalle's invariance principle (Khalil and Grizzle (2002)), the trajectory of the algorithm (27)-(30) will be eventually contained in a compact subset of the invariant set

$$\mathcal{I} = \{(\mathbf{p},\boldsymbol{\lambda}) : \dot{U}(\mathbf{p},\boldsymbol{\lambda}^1;\mathbf{p}^*,\boldsymbol{\lambda}^{1*}) = 0\}. \quad (37)$$

Since $\bar{L}(\mathbf{p};\boldsymbol{\lambda}^1)$ is strictly concave in $\boldsymbol{\lambda}^1$, by Proposition 11 in You and Chen (2014) the invariant set $\mathcal{I}$ is a subset of the primal-dual optima of the problem (31)-(33) and its dual, and $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$ for all $(\mathbf{p},\boldsymbol{\lambda}) \in \mathcal{I}$. When the network is a tree, the set $\mathcal{I}$ is a singleton, and obviously the algorithm (27)-(30) converges to the unique primal-dual optimum of the problem (31)-(33) and its dual. In general, for any networks, since the algorithm converges to the compact set $\mathcal{I}$ as $t \to \infty$, there exists a convergence subsequence $\{(\mathbf{p}(t_k),\boldsymbol{\lambda}(t_k))\}_{k=1,2,\ldots}$ with $0 \le t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k \to \infty$, such that $\lim_{k\to\infty}\mathbf{p}(t_k) = \mathbf{p}^\infty$ and $\lim_{k\to\infty}\boldsymbol{\lambda}(t_k) = \boldsymbol{\lambda}^*$ for some $(\mathbf{p}^\infty,\boldsymbol{\lambda}^*) \in \mathcal{I}$. Since the Lyapunov function can be defined in terms of any primal-dual optimum, we choose the Lyapunov function to be $U(\mathbf{p},\boldsymbol{\lambda}^1;\mathbf{p}^\infty,\boldsymbol{\lambda}^{1*})$. Notice that $U \ge 0$ with $U = 0$ only if $\mathbf{p} = \mathbf{p}^\infty$, and $\dot{U} \le 0$ along the trajectory $(\mathbf{p}(t),\boldsymbol{\lambda}(t))$ of the algorithm (27)-(30). By the continuity of U, we have

$$\begin{aligned}&\lim_{t\to\infty} U(\mathbf{p}(t),\boldsymbol{\lambda}^1(t);\mathbf{p}^\infty,\boldsymbol{\lambda}^{1*})\\ &= \lim_{k\to\infty} U(\mathbf{p}(t_k),\boldsymbol{\lambda}^1(t_k);\mathbf{p}^\infty,\boldsymbol{\lambda}^{1*})\\ &= U(\mathbf{p}^\infty,\boldsymbol{\lambda}^{1*};\mathbf{p}^\infty,\boldsymbol{\lambda}^{1*}) = 0.\end{aligned}$$

This implies that $(\mathbf{p}(t),\boldsymbol{\lambda}(t))$ converges to $(\mathbf{p}^\infty,\boldsymbol{\lambda}^*)$, which is a primal-dual optimum of the problem (31)-(33) and its dual.

Theorem 9 does not implies *global* convergence of the algorithm (27)-(30), as its proof requires that the trajectory of the algorithm is contained in the region defined by $-\mathbf{b} \prec \mathbf{p} \prec \mathbf{b}$. Moreover, the convergence is *trajectory-wise* and does not necessarily imply the local stability of the primal-dual optimum $(\mathbf{p}^\infty,\boldsymbol{\lambda}^*)$. We will however show that the convergence point $(\mathbf{p}^\infty,\boldsymbol{\lambda}^*)$ is unique, i.e., independent of the specific trajectories, and is indeed locally stable.

Theorem 10. The primal-dual gradient algorithm (27)-(30) converges to a unique and locally stable primal-dual optimum of the problem (31)-(33) and its dual.

Proof. By Theorem 8 or equation (29), at a convergence point $(\mathbf{p}^\infty,\boldsymbol{\lambda}^*)$, there exist phases $\boldsymbol{\theta}$ with $|\theta_{s_l} - \theta_{d_l}| < \pi/2,\ l \in \mathcal{E}$ such that

$$\bar{f}_i = \sum_{l\in\mathcal{E}} A_{il} p_l = \sum_{l\in\mathcal{E}} A_{il} b_l \sin(\theta_{s_l} - \theta_{d_l}),\ i \in \mathcal{N}. \quad (38)$$

Notice that the above mapping from $\boldsymbol{\theta}$ to $\bar{\mathbf{f}}$ is one to one in the domain defined by $|\theta_i - \theta_j| < \pi/2,\ (i,j) \in \mathcal{E}$; see, e.g., Araposthatis et al. (1981). So, $\mathbf{p}^\infty$ and thus $(\mathbf{p}^\infty,\boldsymbol{\lambda}^*)$ are uniquely determined, independent of specific trajectories of the algorithm (27)-(30). This further implies that the convergence point $(\mathbf{p}^\infty,\boldsymbol{\lambda}^*)$ is locally stable.

Combining Theorems 8-10, we have the following result.

Theorem 11. The following two statements are equivalent:

(1) There exits at least one $\bar{\mathbf{p}} \in \mathrm{Ker}(A)$ such that $-\mathbf{b} \prec A^T(AA^T)^\dagger\bar{\mathbf{f}} + \bar{\mathbf{p}} \prec \mathbf{b}$.
(2) The network of coupled oscillators (1)-(2) has a unique and locally stable synchronization equilibrium with cohesive phases $|\theta_i^0 - \theta_j^0| < \pi/2,\ (i,j) \in \mathcal{E}$.

Theorem 11 states the synchronization condition elegantly as the existence of solution for a system of linear equations, and is a complete characterization of the condition for synchronization of coupled oscillator network in the region defined by $|\theta_i - \theta_j| < \pi/2,\ (i,j) \in \mathcal{E}$.

Similar to Corollary 7, we have the following sufficient condition for synchronization equilibrium if choosing $\bar{\mathbf{p}} = 0$.

Corollary 12. The network of coupled oscillators (1)-(2) has a unique and locally stable synchronization equilibrium with cohesive phases $|\theta_i^0 - \theta_j^0| < \pi/2,\ (i,j) \in \mathcal{E}$, if

$$||(\mathrm{diag}(\mathrm{b}))^{-1} A^T(AA^T)^\dagger\bar{\mathbf{f}}||_\infty < 1. \quad (39)$$

This sufficient condition is exactly one condition given in Dorfler et al. (2013).

Notice that the synchronization equilibrium of the coupled oscillator network is locally stable. An important question is to characterize its region of attraction, which has important implication in applications to, e.g., the power network. We will explore the Lyapunov function (34) and its convexity to investigate this question in future work.

To recapture, the above conditions for synchronization of the coupled oscillator network (1)-(2) are carried over from the conditions for synchronization solution of the KKT system for the problem (12)-(14). We have reduced the hard problem of synchronization of coupled oscillators to a simple problem of verifying solution of a system of linear equations, by identifying the network system dynamics as a distributed partial primal-dual gradient algorithm for solving a well-defined convex optimization problem and its dual.

4. IMPLICATION ON CONVEX RELAXATION

Consider the following optimization problem:

$$\min_{\mathbf{d},\boldsymbol{\theta}} \quad \sum_{i\in\mathcal{N}} C_i(d_i) \quad (40)$$

$$\text{subject to} \quad f_i = d_i + \sum_{l\in\mathcal{E}} A_{il} b_l \sin(\theta_{s_l} - \theta_{d_l}),\ i \in \mathcal{N}, \quad (41)$$

where the "physical" constraint (41) captures the force balance that should hold at an equilibrium. The problem (40)-(41) looks a more natural problem to study than the problem (12)-(14), as it captures directly nonlinear coupling between the oscillators. In the power network application, for instance, the problem (40)-(41) corresponds to an optimal power flow problem with nonlinear branch flows, i.e., without assuming small phase deviation as in usual DC power flow approximation.

The problem (40)-(41) is nonconvex, even if the phases are constrained to $|\theta_{s_l} - \theta_{d_l}| \le \pi/2,\ l \in \mathcal{E}$. Notice that the problem (12)-(14) is a convex relaxation of the problem (40)-(41), and at its optimum the constraint (41) is satisfied when solved using the algorithm (27)-(30).

Theorem 13. The network dynamics (1)-(2) in the region defined by $|\theta_i - \theta_j| < \pi/2,\ (i,j) \in \mathcal{E}$ is a distributed algorithm for solving the problem (40)-(41).

Notice that an optimum of the problem (12)-(14) may not be an optimum of the problem (40)-(41). Theorem 13 thus has an interesting implication: a non-convex problem may be solved through solving its convex relaxation using a *carefully chosen algorithm*. This kind of exact convex relaxation is a bit different from the "conventional" exact relaxation where the optimum of the convex problem is always a feasible point of the original non-convex problem. Physically, this confirms an insight that a physical system usually solves a convex problem (e.g., the problem (12)-(14)) even though it may have a non-convex representation (e.g., the problem (40)-(41)). Even though the above implication on exact convex relaxation is based on the result when the phases are restricted to $|\theta_{s_l} - \theta_{d_l}| < \pi/2,\ l \in \mathcal{E}$, we expect that it holds generally and will further investigate it in future work.

5. CONCLUSION

We have taken a new approach to investigate synchronization in the coupled oscillator network, by identifying the network system dynamics as a distributed primal-dual gradient algorithm for solving a well-defined convex optimization problem and its dual. This new approach reduces the hard problem of synchronization of coupled oscillators to a simple problem of verifying synchronization solution of a system of linear equations, and leads to a complete characterization of synchronization condition for the coupled oscillator network in an interesting and practically important region. Our synchronization condition is stated elegantly as the existence of solution for a system of linear equations, of which one best existing synchronization condition is a special case of sufficient condition. We have also formulated a non-convex optimization problem with the force balance constraints for which the afore convex optimization problem is relaxation, and showed that the coupled oscillator system is also a distributed algorithm for solving this non-convex problem. This has interesting implication on exact convex relaxation, and confirms the insight that a physical system usually solves a convex problem even though it may have a non-convex representation.

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