Economic equilibrium and optimization problems using GAMS
Notes 2: from optimization (NLP) to complementarity (MCP)

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Tools of Economic Analysis

(1)   Analytical theory models

(2)   Econometric estimation and testing.

(3)   Simulation modeling - complement to (1) and (2)

   (A) greatly extends the reach of theory to problems that are
       analytically intractable

   (B) extends the economic usefulness of econometrics allowing
       counter-factuals using estimation for calibration

(4) Two ways of formulating economic models

    (A) as an constrained optimization problem

    (B) as an economic equilibrium problem: square system of equations/inequalities and unknowns

(5) Limitations of analytical theory

"Many branches of both pure and applied mathematics are in great need of computing instruments to break the present stalemate created by the failure of the purely analytical approach to nonlinear problems"

    --- John Von Neumann, 1945

Analytical methods quickly become intractable

(1) functions or equation systems have no closed-form solution

(2) large dimensionality (# of equations and unknowns)

(3) correct model consists of non-linear weak inequalities

(6)   Responses to difficulties

(1) stick to analytical methods, eliminate difficulties by restrictive assumptions (even if eliminating the most interesting parts of the problem)

(2) simulate the model you really want to solve

(3) make an analytical model-of-the-model (e.g., partial equilibrium) and then simulate the richer model

(7)   Structure of a typical model

Economic models are based on the assumption of optimizing
agents: consumers, firms, governments

But, generally the model itself cannot be written as a simple
constrained optimization problem

Example 1: two households with different preferences/incomes
Example 2: two-firm duopoly model

An economic model typically embodies optimization at the level of
the agent, the model becomes an nxn equilibrium problem

Household or firm represented by constrained optimization problem (non-linear programming problem).

Karush-Kuhn-Tucker theorem: converts NLP into a square system of equations/unknowns, including "slack" variables (Lagranean multipliers).

Implicit function theorem => square system can be solved for explicit functions: endogenous variables each as a function of system parameters.

More often, two-step procedure: KKT functions used to derive cost / expenditure / or indirect utility or profit functions.

Envelop theorem + Shepard's lemma used to derive goods and factor demand functions, supply functions, etc.

(8) Dimensionality, inequalities, bounded variables

Dimensionality, non-linearity, and simultaneity make models hard to solve analytically past 2 equations in 2 unknowns

Example: 2 factor, 2 good, 2 country Heckscher-Ohlin model

Economics variables are typically bounded (e.g., prices and quantities are non-negative) and economic equilibrium conditions are weak inequalities.

Example: what goods produced, technologies used?
Example: what trade links are active in equilibrium?
Example: do emissions permits have a positive price?

Economics is often sacrificed to ensure a strictly interior solution.

(9) Complementarity: KKT conditions from underlying NLP:

with weak inequality constraints, including non-negative restrictions on economic variables (prices and quantities)

Equilibrium conditions are weak inequalities

Each inequality is associated with a particular variable, called the complementary variable.

If the equation holds as an equality in equilibrium, then the complementary variable is generally strictly positive.

If the equation holds as a strict inequality in equilibrium, the complementary variable is zero.

(10)   GAMS solvers: two ways of formulating economic models

   (A)   NLP: non-linear programming
              constrained optimization

   (B)   MCP: mixed complementarity problem
              square system of equations/inequalities and unknowns
              matched inequalities and variables

   (C)   MPEC: mathematical programming with equilibrium
              constraints: NLP + MCP constraint set

   Matching of equations/inequalities and the direction of the
       inequalities must come from the modeler in accordance with
       economic theory.

Production and Complementarity: the optimality condition for a firm is an inequality complementary with its output variable. c(...) = unit (or marginal) cost function.

Maximize profits $\pi = pX - c(...)X \qquad s.t. \qquad X \geq 0$

Using the Karush-Kuhn-Tucker theorem, there exits a non-negative constant µ such that the necessary conditions for the maximum

$$p - c(...) + \mu = 0 \qquad \mu'X = 0 \qquad \mu \geq 0$$

If marginal cost is strictly greater than price in equilibrium, then output is zero.

In GAMS, the optimality condition is written as an inequality and specifies the complementary variable. The multiplier µ is created "behind the scenes" by GAMS. Inequalities in GAMS are always written as greater than or equal to.

$$c(...) \geq p \quad \perp \quad X \quad \text{"the weak inequality is complementary to } X\text{"}$$

Though the modeler does not specify the slack variable μ, it will appear in the GAMS output file under the name "marginal". The value of marginal equals μ.

It follows from the KKT conditions that if the variable X is strictly positive in the solution, then μ = 0. If X = 0 in the solution, then μ is strictly positive.

μ = $c(...)$ - p, and the slack variable μ measures the amount by which marginal cost exceeds p in equilibrium.

Alternatively, μ is the difference between the left and right-hand sides of the weak inequality.

Market-Clearing and Complementarity: market clearing requires that the value (price times quantity) supplied in a market equal the value of demand.

Let D denote the demand for a good while X denotes supply.

$pX = pD$ there are three possibilities

$X = D$ $p > 0$ usual "interior" solution

$X > D$ $p = 0$ excess supply, X is a "free good"

$X < D$ $p = 0$ *violates* standard assumption: excess demand will cause the price to rise

We should model a market-clearing condition as a weak inequality complementary with the price of the good.

$X \geq D$ $\perp$ p "the weak inequality is complementary to p"

When GAMS is given this weak inequality and complementary
variable, it will create equation with a slack variable analogous to
the procedure for the pricing equation.

$$X - D + \mu = 0 \qquad \mu'p = 0 \qquad p \geq 0 \qquad \mu \geq 0$$

$$X = D \implies p > 0 \qquad X > D \implies p = 0$$

If the inequality is strict in equilibrium, there is excess supply in
equilibrium and the price is zero: X is a free good.

In that case, the auxiliary variable μ, called the "marginal" in the
GAMS listing file, has a positive value equal to the supply-
demand imbalance in equilibrium: $\mu = X - D$.

(12)   Simple supply-demand problem illustrating complementarity

Supply and demand model of a single market. Two equations: supply and demand. Two variables: price and quantity.

Economic equilibrium problems are represented as a system of n equations/inequalities in n matched unknowns.

Supply of good X with price P.   The supply curve exploits the firm's optimization decision, $P = MC.$

   $MC \geq P$ with the complementarity condition that $X \geq 0$

The *price* equation is complementary with a *quantity* variable.

Suppose that COST = $aX + (b/2)X^2$.

Marginal cost is then given by $MC = a + bX$.

$a + bX \geq P$ complementary with $X \geq 0$.

Optimizing consumer utility for a given income and prices will yield a demand function of the form $X = D(P, M)$ where M is income.

$X \geq D(P, M)$ with the complementary condition that $P \geq 0$.

The *quantity* equation is complementary with a *price* variable.

We will suppress income and assume a simple function:

$D(P) = c - dP$ where $c > 0$, $d > 0$.[1]

$X \geq c - dP$ complementary with $P \geq 0$.

How do we know which inequality is associate with which variable and the direction of the inequality?

Economic theory tells you which variable must be associated with which inequality and which way the inequality goes.

_____

[1]This demand function can be derived as the solution to a constrained optimization problem in which the consumer has a quasi-linear utility function of the form $U = \alpha X - \beta X^2 + Y$ and budget constraint $M = p_x X + p_y Y$

Use of slack variables to convert weak inequalities to equalities.

Introduce non-negative variables S1, S2 $\geq$ 0

    MC(X) - P - S1 = 0

    S1*X = 0
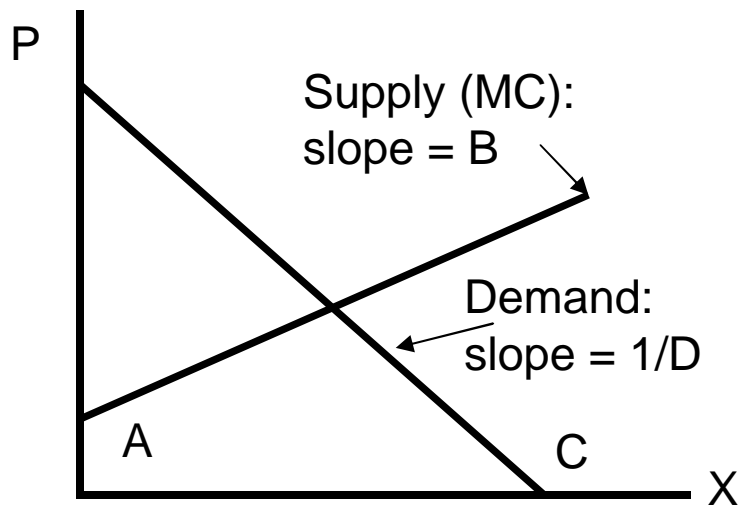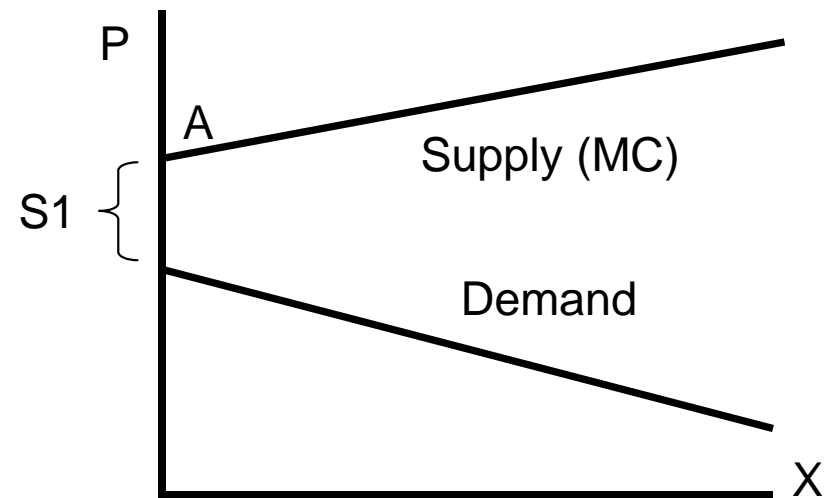
    X - D(P) - S2 = 0

    S2*P = 0

Four equations in four unknowns. Note: S1, S2 give the *imbalances* in their corresponding equations in equilibrium.

BUT, there are still the non-negativity constraints to worry about. This is done (I think) in the solution algorithm (discussed later).
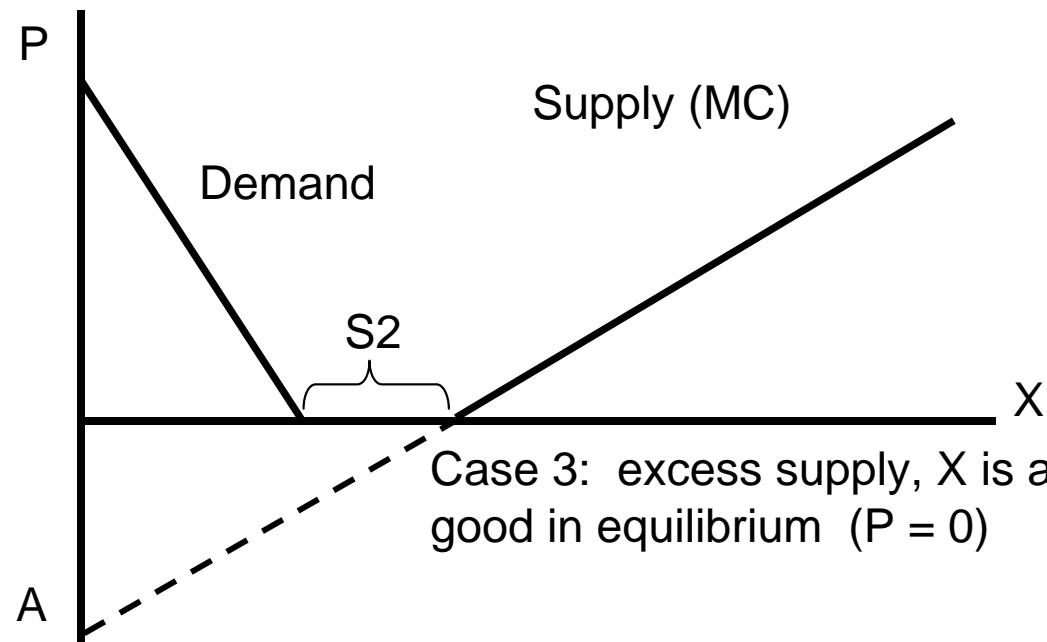
Figure 1: Three outcomes of partial equilibrium example

P

Supply (MC):
slope = B

Demand:
slope = 1/D

A

C

X

Case 1: interior solution

P

A

S1

Supply (MC)

Demand

X

Case 2: X is too expensive, not
produce in equilibrium (X = 0)

P

Supply (MC)

Demand

S2

X

A

Case 3: excess supply, X is a free
good in equilibrium (P = 0)

(12)   Coding an economic equilibrium problem in GAMS

First, comment statements can be used at the beginning of the code, preceded with a *, in the *first column* of a line.

```
$TITLE: M2-1.GMS  introductory model using MCP and MPEC
* simple supply and demand model
```

Begin a series of declaration and assignment statements.

```
PARAMETERS
 A      intercept of supply on the P axis (MC at Q = 0)
 B      slope of supply:  this is dP over dQ
 C      demand on the Q axis (demand at P = 0)
 D      (inverse) slope of demand, dQ over dP;
```

Parameters must be assigned values before the model is solved

```
A = 2;   C = 6;   B = 1;   D = 1;
```

Declare a list of variables. They are restricted to be non-negative to make any economic sense, so declaring them as "nonnegative variables" tells GAMS to set lower bounds of zero.

```
NONNEGATIVE VARIABLES
  P      price of good X
  X      quantity of good X;
```

Now we similarly declare a list of equations. Name not otherwise in use or, of course, a keyword.

```
EQUATIONS
  SUPPLY       supply relationship (mc cost ge price)
  DEMAND       quantity demanded as a function of price;
```

Specify equations. Format: equation name followed by two periods
   The equation is written with =G= for "greater than or equal to".

```
SUPPLY..     A + B*X   =G=   P;

DEMAND..     X   =G=   C + D*P;
```

Declare a model. Keyword *model,* followed by a model name.

Then a "/" followed by a list of the equation names, each ends with a
   period followed by the name of the complementary variable.

```
MODEL EQUIL /SUPPLY.X, DEMAND.P/;
```

Tell GAMS to solve the model and what solver is needed.

```
SOLVE EQUIL USING MCP;
```

This example uses parameter values which generate an "interior solution", meaning that both X and P are strictly positive.

Case 2: the good, or a particular way to produce or obtain a good (e.g., via imports) is too expensive relative to some alternative: production or trade activity is not used in equilibrium: X = 0.

```
A = 7;
SOLVE EQUIL USING MCP;
```

Case 3: The final possibility is that a good or factor of production may be so plentiful that it commands a zero price in equilibrium

```
A = -7;
SOLVE EQUIL USING MCP;
```

(13)   Reading the output (file name G1.LST)

GAMS stores four values for a varlable. *LOWER* and *UPPER* are
    bounds on the variables. Declaring a *NONNEGATIVE VARIABLE*
    sets the lower bound at 0 (.) and upper bound at +inf.

The *LEVEL* is the solution value of the variables.

MARGINAL indicates the degree to which the equation
    corresponding to the variable is out of equality.

For *P* (price), the equation is *DEMAND* and the value of the marginal
    is supply minus demand.

For *X* (quantity), the equation is *SUPPLY* and the value of the
    marginal is the excess of marginal cost over price.

Variables that have *positive values* in the solution should have *zero marginals*.

Variables that have *zero values* in the solution should have *positive marginals*.

Here is the benchmark case.

```
                    LOWER        LEVEL        UPPER        MARGINAL

---- VAR P            .          4.000        +INF
---- VAR X            .          2.000        +INF
```

Case 2: This is the zero-output case. The price equation holds, but the quantity equation is slack. The marginal of 1.0 indicates that, at the solution, marginal cost exceed price by 1.0.

```
                     LOWER        LEVEL        UPPER       MARGINAL

---- VAR P             .          6.000        +INF           .
---- VAR X             .            .          +INF         1.000
```

Case 3: This is the free-good case. Now the price equation is slack, and the marginal of 1.0 indicates that, at the solution, supply exceeds demand by 1.0.

```
                     LOWER        LEVEL        UPPER       MARGINAL

---- VAR P             .            .          +INF         1.000
---- VAR X             .          7.000        +INF           .
```

(14) Example of the use of MPEC: set an endogenous tax rate that maximizes tax revenue.

Declare two (unbounded) variables and equations.

```
VARIABLES
 T       tax rate on marginal cost
 TREV    tax revenue = MC*T*X;


EQUATIONS
 SUPPLY2 new supply function incorporating endogenous tax
 OBJ      objective function is tax revenue;

OBJ..          TREV =E= (A + B*X)*T*X;

SUPPLY2..    (A + B*X)*(1+T) =G= P;

OPTION MPEC = NLPEC;

MODEL TREVENUE /OBJ, SUPPLY2.X, DEMAND.P/;
SOLVE TREVENUE USING MPEC MAXIMIZING TREV;
```
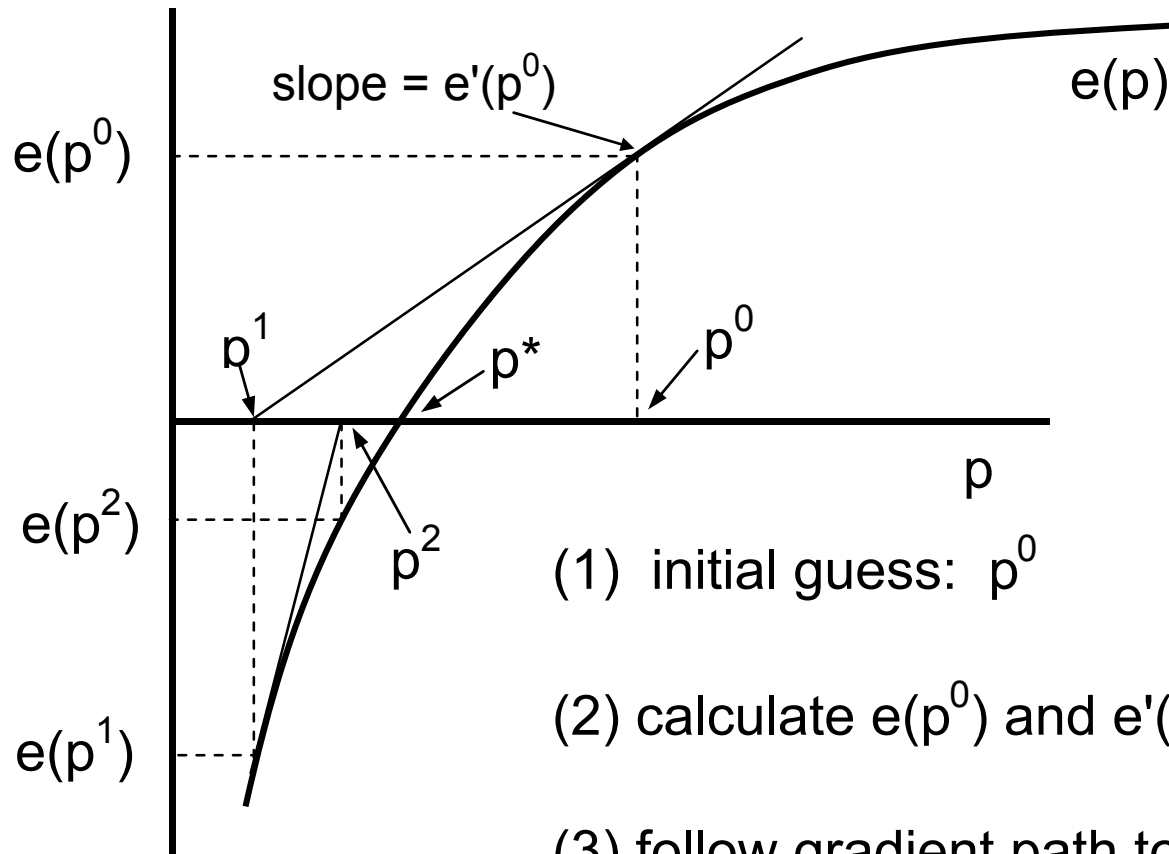
	LOWER	LEVEL	UPPER	MARGINAL
---- VAR P	.	5.000	+INF	EPS
---- VAR X	.	1.000	+INF	.
---- VAR T	-INF	0.667	+INF	.
---- VAR TREV	-INF	2.000	+INF	.

Illustration of gradient method: find the zero of the excess supply function: $e(p^*) = s(p^*) - d(p^*) = 0$

slope = $e'(p^0)$

$e(p)$

$e(p^0)$

p^1

p^*

p^0

p

iteration rule:
$$p^{i+1} = p^i - [e'(p^i)]^{-1}e(p^i)$$

$e(p^2)$

p^2

(1) initial guess: p^0

$e(p^1)$

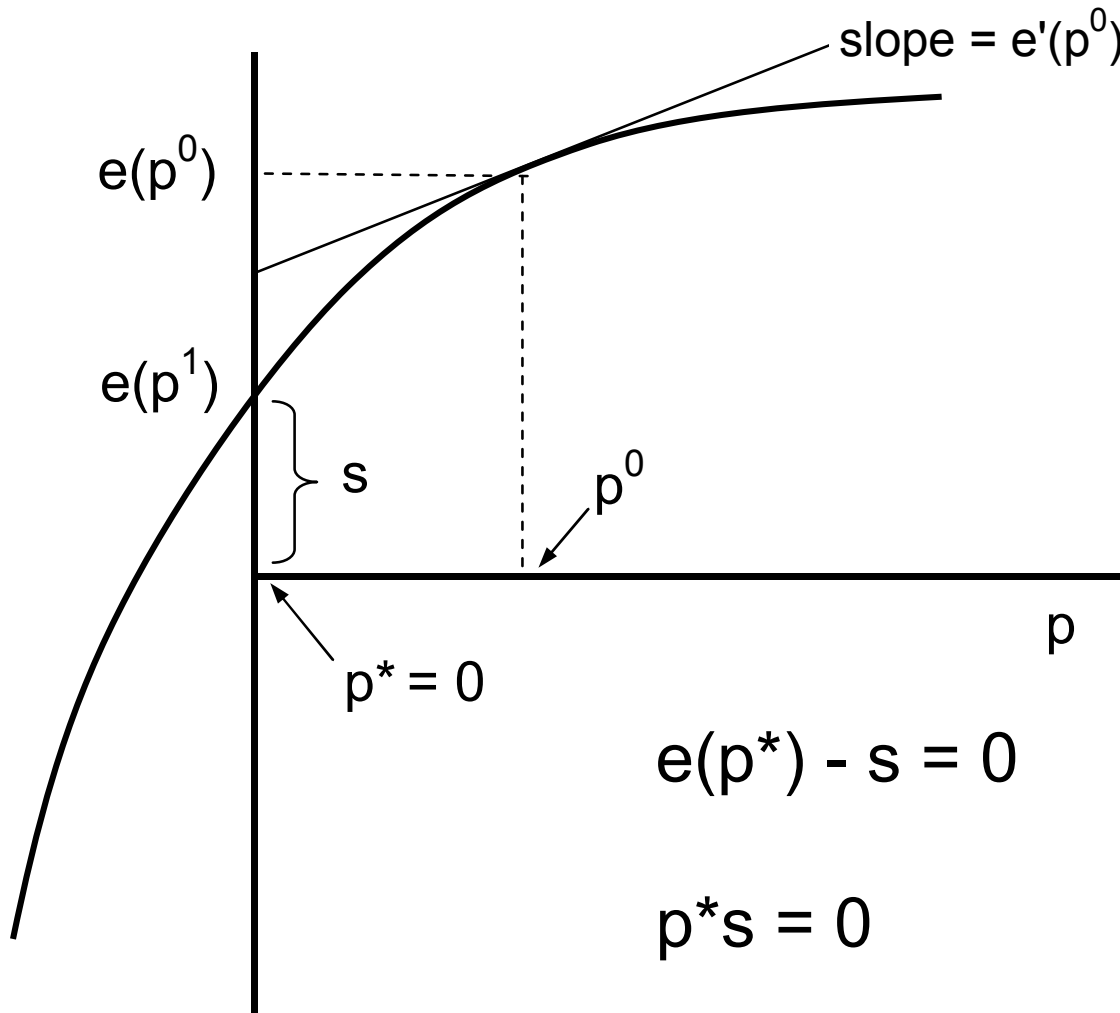(2) calculate $e(p^0)$ and $e'(p^0)$ (exit if $|e(p^0)| < \varepsilon$)

(3) follow gradient path to $e = 0$

(4) new guess p1 implicitly given by
$$e(p^0)/(p^0 - p^1) = e'(p^0)$$

$$p^{i+1} = p^i - [e'(p^i)]^{-1}e(p^i) \quad \text{(generalizes to nxn)}$$

Illustration of gradient method: find the zero of the
 excess supply function: $e(p^*) - s = 0$

slope = $e'(p^0)$

$e(p^0)$

$e(p^1)$

s

p^0

p

$p^* = 0$

$e(p^*) - s = 0$

$p^*s = 0$

$s , p \geq 0$

Economic equilibrium and optimization problems using GAMS
Continuation of Notes 2
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Maximization of utility subject to a linear budget constraint

Illustrates the use of the GAMS NLP and MCP solvers

NLP non-linear programming
MCP mixed complementarity problem

Cobb-Douglas utility function with linear budget constraint

$$U = \left(\frac{X_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{X_2}{\alpha_2}\right)^{\alpha_2} \qquad M = p_1 X_1 + p_2 X_2 \qquad \alpha_1 + \alpha_2 = 1$$

Result: C-D exponents are expenditure shares: $\qquad \alpha_1 = \dfrac{p_1 X_1}{M}$

- Option 1: direct solution as an NLP problem

- Option 2a: use KKT to formulated as an MCP, "primal" version

$$\max \quad U = \left(\frac{X_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{X_2}{\alpha_2}\right)^{\alpha_2} + \lambda(M - p_1X_1 - p_2X_2)$$

$$\frac{\partial U}{\partial X_1} = \left(\frac{X_1}{\alpha_1}\right)^{\alpha_1 - 1}\left(\frac{X_2}{\alpha_2}\right)^{\alpha_2} - \lambda p_1 \leq 0 \qquad \perp \qquad X_1$$

$$\frac{\partial U}{\partial X_2} = \left(\frac{X_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{X_2}{\alpha_2}\right)^{\alpha_2 - 1} - \lambda p_2 \leq 0 \qquad \perp \qquad X_2$$

$$\frac{\partial U}{\partial \lambda} = M - p_1X_1 - p_2X_2 \geq 0 \qquad \perp \qquad \lambda$$

Motivated by the *implicit function theorem*, we can solve for "Marshallian" or "uncompensated" demand functions

$$X_1 = D_1(p_1, p_2, M) = \frac{\alpha_1 M}{p_1} \qquad X_1 = \frac{\alpha_2 M}{p_2} \qquad \text{How?}$$

Divide first FOC by the second; $\qquad \dfrac{X_1}{X_2} = \dfrac{\alpha_1}{\alpha_2} \dfrac{p_2}{p_1} \qquad p_2 X_2 = \dfrac{\alpha_2}{\alpha_1} p_1 X_1$

Use to replace $p_2 X_2$ in budget: $\qquad M - p_1 X_1 - \dfrac{\alpha_2}{\alpha_1} p_1 X_1 = 0$

Solve this equation for X_1: $\qquad X_1 = \dfrac{\alpha_1}{\alpha_1 + \alpha_2} \dfrac{M}{p_1} = \dfrac{\alpha_1 M}{p_1}$

$\qquad$ since $\alpha_1 + \alpha_2 = 1$

Note we can substitute these back into the utility function to get the indirect utility function.

$$U \;=\; V(p_1, p_2, M) \;=\; p_1^{-\alpha_1} p_2^{-\alpha_2} M$$

V is termed a *value function*; that is, it is the *maximized* value of the Lagrangean function for given values of the parameters p_1, p_2, and M.

More on this later.

- Option 2b: use KKT to formulated as an MCP, "dual" version

$$
\min \ p_1 X_1 + p_2 X_2 + \mu \left(U - \left(\frac{X_1}{\alpha_1} \right)^{\alpha_1} \left(\frac{X_2}{\alpha_2} \right)^{\alpha_2} \right)
$$

$$
\frac{\partial E}{\partial X_1} = p_1 - \mu \left(\frac{X_1}{\alpha_1} \right)^{\alpha_1 - 1} \left(\frac{X_2}{\alpha_2} \right)^{\alpha_2} \leq 0
$$

$$
\frac{\partial E}{\partial X_2} = p_2 - \mu \left(\frac{X_1}{\alpha_1} \right)^{\alpha_1} \left(\frac{X_2}{\alpha_2} \right)^{\alpha_2 - 1} \leq 0
$$

$$
\frac{\partial E}{\partial \mu} = U - \left(\frac{X_1}{\alpha_1} \right)^{\alpha_1} \left(\frac{X_2}{\alpha_2} \right)^{\alpha_2} \leq 0
$$

Motivated by the *implicit function theorem*, we can solve for "Hicksian" or "compensated" demand functions

$$X_i = H_i(p_1, p_2, U)$$

Divide first FOC by the second;

$$\frac{X_1}{X_2} = \frac{\alpha_1}{\alpha_2} \frac{p_2}{p_1} \qquad X_2 = \frac{\alpha_2}{\alpha_1} \frac{p_1}{p_2} X_1$$

Substitute this into constraint:

$$U - \left(\frac{X_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{X_1 p_1}{\alpha_1 p_2}\right)^{\alpha_2} = \frac{X_1}{\alpha_1}\left(\frac{p_1}{p_2}\right)^{\alpha_2} = 0$$

$$X_1 = \alpha_1 p_1^{-\alpha_2} p_2^{\alpha_2} U = \alpha_1 p_1^{\alpha_1 - 1} p_2^{\alpha_2} U \qquad X_2 = \alpha_2 p_1^{\alpha_1} p_2^{\alpha_2 - 1} U$$

Now construct the cost function: the minimum cost of buying U units of utility. This is called the "expenditure function" in microeconomics.

$$E(p_1, p_2, U) \;=\; p_1 X_1 + p_2 X_2 \quad \text{Substituting from the previous eq}$$

$$=\; \alpha_1 p_1^{\alpha_1} p_2^{\alpha_2} U \;+\; \alpha_2 p_1^{\alpha_1} p_2^{\alpha_2} U \;=\; p_1^{\alpha_1} p_2^{\alpha_2} U$$

Next, note that the partial derivative of the expenditure function with respect to a price is the (Hicksian) demand for that good.

$$\frac{\partial E(p_1, p_2, U)}{\partial p_1} \;=\; \alpha_1 p_1^{\alpha_1 - 1} p_2^{\alpha_2} U \;=\; X_1 \;=\; H_1(p_1, p_2, U)$$

Due to Shepard's lemma, following from the envelop theorem.

Envelop theorem: in our case, E is called a *value function*. Specfically, it is the *minimized* value of expenditure given the parameters prices and utility.

Consider a function *F(Y,β)* where *Y* is a variable(s) and β is a parameter(s). In our case *F* is the Lagranean function. The value function *V* is

$$V(\beta) \;=\; \mathop{\min}_{Y} F(Y, \beta) \qquad \text{=> for a given β, } Y \text{ is chosen such that}$$

$$\frac{\partial F}{\partial Y} \;=\; 0 \quad \text{(first-order condition)} \qquad \text{it then follows that}$$

$$\frac{dV}{d\beta} \;=\; \frac{\partial F}{\partial \beta} \;+\; \frac{\partial F}{\partial Y}\frac{\partial Y}{\partial \beta} \;=\; \frac{\partial F}{\partial \beta}$$

In our case, $Y = [X_1, X_2, \mu], \quad \beta = [p_1, p_2, U]$

$$F(Y, \beta) = p_1 X_1 + p_2 X_2 + \mu\left(U - \left(\frac{X_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{X_2}{\alpha_2}\right)^{\alpha_2} \right)$$

$$E(p_1, p_2, U) = \min_X \; p_1 X_1 + p_2 X_2 + \mu\left(U - \left(\frac{X_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{X_2}{\alpha_2}\right)^{\alpha_2} \right)$$

$$\Rightarrow \; \frac{\partial F}{\partial X_i} = 0$$

$$\left[\frac{dE}{dp_i}\right] = X_i = \frac{\partial F}{\partial p_i} = \frac{dF}{dp_i} = X_i + \left[\frac{\partial F}{\partial X_1}\frac{dX_1}{dp_i} + \frac{\partial F}{\partial X_2}\frac{dX_2}{dp_i}\right]$$

$$= 0$$

Cost and factor-demand functions for firms.

The expenditure function is just a specific case of a cost function.

It is important for us to note that the exact same procedure allows us to derive a firm's cost function and, from that, to derive the firm's demands for factors of production.

Let X be an output produced from K (capital) and L (labor), which have factor prices p_k and p_l respectively.

The firm's cost function is given by

$$X = \left(\frac{K}{\alpha_1} \right)^{\alpha_1} \left(\frac{L}{\alpha_2} \right)^{\alpha_2} \qquad \alpha_1 + \alpha_2 = 1 \text{ (constant returns to scale)}$$

$$C(p_k, p_l, X) = \min_{K,L} \; p_k K + p_l L \;+\; \mu\left(X - \left(\frac{K}{\alpha_1}\right)^{\alpha_1}\left(\frac{L}{\alpha_2}\right)^{\alpha_2}\right)$$

$$C(p_l, p_k, X) = p_k^{\alpha_1} p_l^{\alpha_2} X = c(p_k, p_l) X$$

Shepard's lemma then gives the firm's demands for K and L conditional on output.

$$\frac{dC(p_k, p_l, X)}{dp_k} = \alpha_1 p_k^{\alpha_i - 1} p_l^{\alpha_2} X = K = D_k(p_k, p_l, X)$$

Note that we could also write this as

$$K = \alpha_1 p_k^{\alpha_i - 1} p_l^{\alpha_2} X = [(\alpha_1 p_k^{\alpha_1} p_l^{\alpha_2})/p_k] X = \alpha_1 [c(p_k, p_l)/p_k] X$$

Why should we care?

In even fairly simple general-equilibrium models, we have multiple
optimizing agents: firms and households.

We need to do the optimization at the level of the firm and household,
deriving the cost and expenditure functions, and then put all the
separate optimization conditions into a model.

We need to derive market-clearing (supply = demand) conditions for
goods and factors. Here we just need to apply Shepard's lemma
to the cost functions we have just derived to get the demands for
goods and factors.