Economics 8858: Simulation modeling in microeconomics

Exercise 1: Cobb-Douglas production and cost functions, calibration

A consumer has a Cobb-Douglas utility function and a linear budget constraint, with income *M*.

$$U = \left(\frac{X_1}{\alpha}\right)^{\alpha}\left(\frac{X_2}{1-\alpha}\right)^{1-\alpha} \qquad M = p_1X_1 + p_2X_2$$

(1) show that the optimal shares of income spent on X_1 and X_2 are given by α and $(1 - \alpha)$ respectively; i.e.,

$$\alpha = \frac{p_1X_1}{M}$$

(2) derive the Marshallian or uncompensated demand functions for X_1 and X_2 :
$X_i = D_i(p_1, p_2, M)$

(3) derive the Hicksian or compensated demand functions for X_1 and X_2 :
$X_i = H_i(p_1, p_2, U)$

(4) derive the expenditure (cost) function, which gives the minimum expenditure necessary at prices p_1 and p_2 to buy one unit of utility.

$$e = e(p_1, p_2) \qquad \text{And} \qquad E = e(p_1, p_2)U$$

(5) demonstrate Shepard's lemma: the derivative of the expenditure function with respect to the price of good i gives the Hicksian demand for good i:

$$X_i = \frac{\partial e(p_1, p_2)}{\partial p_i}U = H(p_1, p_2, U)$$

(6) show that in this Cobb-Douglas case that

$$X_1 = H(p_1, p_2, U) = \alpha e(p_1, p_2)/p_1 = \alpha p_u/p_1$$

where *pu* is the "price" (cost) of buying one unit of utility.

(7) derive the indirect utility function, which gives the (maximum) level of utility reached at prices p_1 and p_2 and income *M*.

$$V = V(p_1, p_2, M)$$

(8) demonstrate Roy's identity:

$$X_i = D_i(p_1, p_2, M) = -V_{p_i}/V_m$$

Note on scaling parameters: suppose that the utility function is given by:

$$U = X_1^{\alpha}X_2^{1-\alpha} \quad \text{then the expenditure function cannot be} \quad e(p_1, p_2) = p_1^{\alpha}p_2^{1-\alpha}$$

Proof: let prices equal one. Then the utility function says that one unit of both goods together produce one unit of utility. The expenditure function says that one unit of utility has a cost of one (each good's price is one).

But both cannot be true. If the utility function is true, then the cost of one unit of utility must be equal to two with prices equal to one. One of these functions must have a scaling parameter.

(9) Derive the expenditure function for the utility function $U = X_1^{\alpha}X_2^{1-\alpha}$

Setting parameters of production and cost functions so that they are consistent with initial data is referred to as "calibration". Of course, the initial data must be "micro-consistent" in that the value of output must equal the sum of the values of the inputs.

Consider a generic Cobb-Douglas production function: X produced from capital (K) and labor(L)

$$X = AL^{\alpha}K^{1-\alpha}$$

Let X^0, L^0, K^0, p_X^0, p_L^0, p_K^0 denote the initial values of quantities and prices observed in the data. Assume that these are micro-consistent, meaning that they satisfy adding up:

$$p^0X^0 = p_L^0L^0 + p_K^0K^0$$

(10) Show that the correctly calibrated production, cost, and factor-demand functions for the Cobb-Douglas functional form are:

$$X = A\left(\frac{L}{\alpha}\right)^{\alpha}\left(\frac{K}{1-\alpha}\right)^{1-\alpha} \qquad A = (p_L^0)^{\alpha}(p_K^0)^{1-\alpha}/p_X^0 \qquad \alpha = \frac{p_L^0L^0}{p_X^0X^0}$$

$$c(p_L, p_k) = p_X^0 (p_L/p_L^0)^{\alpha} (p_K/p_K^0)^{1-\alpha} \qquad (= p_L^{\alpha} p_K^{1-\alpha}/A)$$

$$L_x = \alpha(p_X/p_L)X \qquad K_x = (1-\alpha)(p_X/p_K)X$$

Note that the factor demand equations do not depend on any initial prices.

Verify that these are correct using the numerical example:

$$p_L^0 = 1.5, \quad L_X^0 = 50, \quad p_K^0 = 0.5, \quad K_X^0 = 50 \quad p_X^0 = 2, \quad X^0 = 50$$