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Lecture 3: Duality, Excess Demand and Offer Functions

The national product or national revenue function

$$r(p,v) = \max_{x} \; p'x \quad s.t. \quad (x,v) \; feasible$$

Properties of r(p,v)

(1) homogeneous of degree 1 in p

(2) homogeneous of degree 1 in v

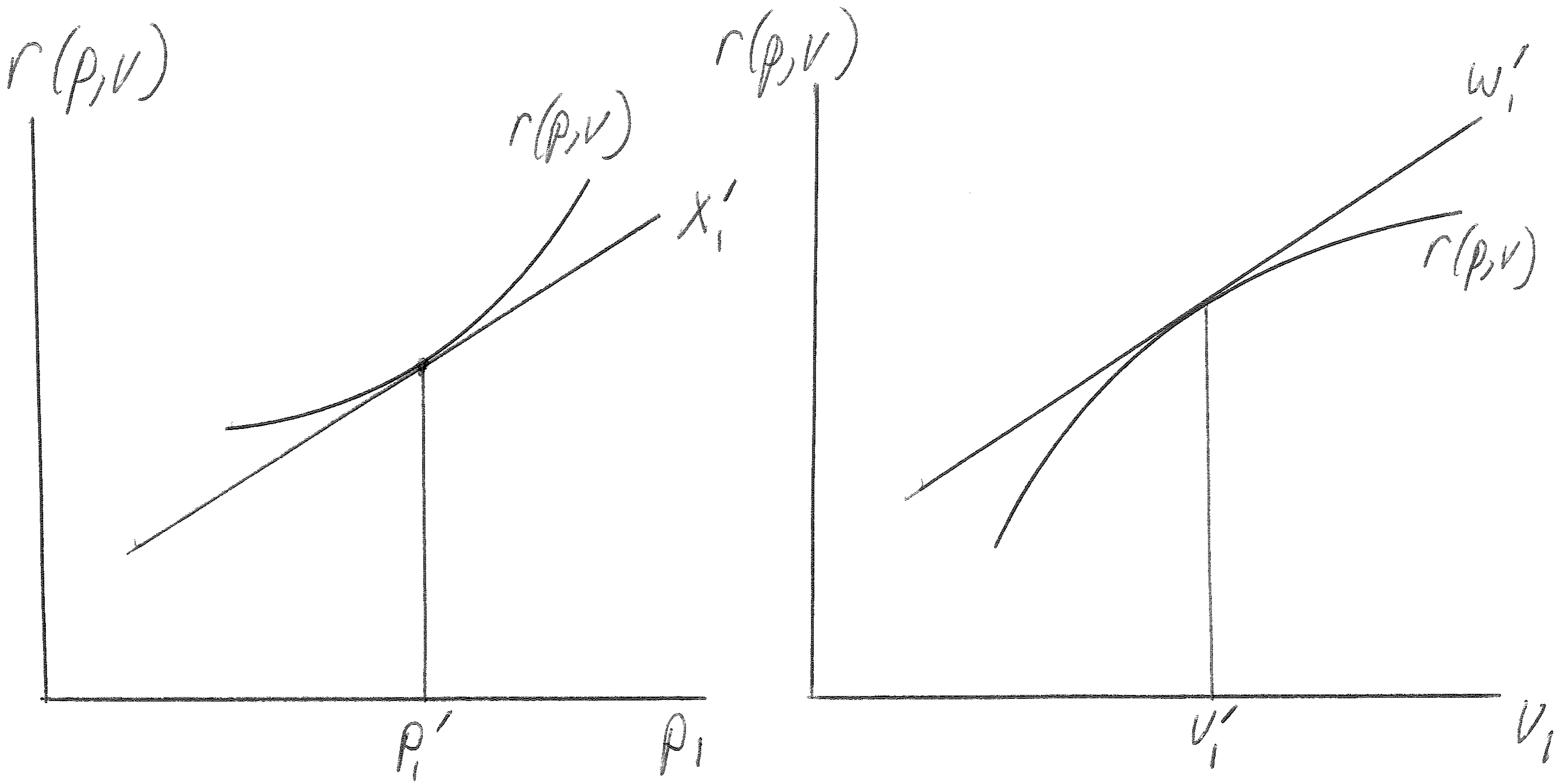
(3) convex in p

Suppose that output x' is chosen at price p'

As we change p, we can at minimum hold x constant. This gives us a linear r(p,v). But by substitution we can always do better. Thus f(p,v) lies on or above the line p'x' except at p'.

(4) concave in v

As we change v, the very best we could do is hold w constant. But with diminishing marginal products, r(p,v) will generally lie below the tangent (where the tangent holds the factor price constant).



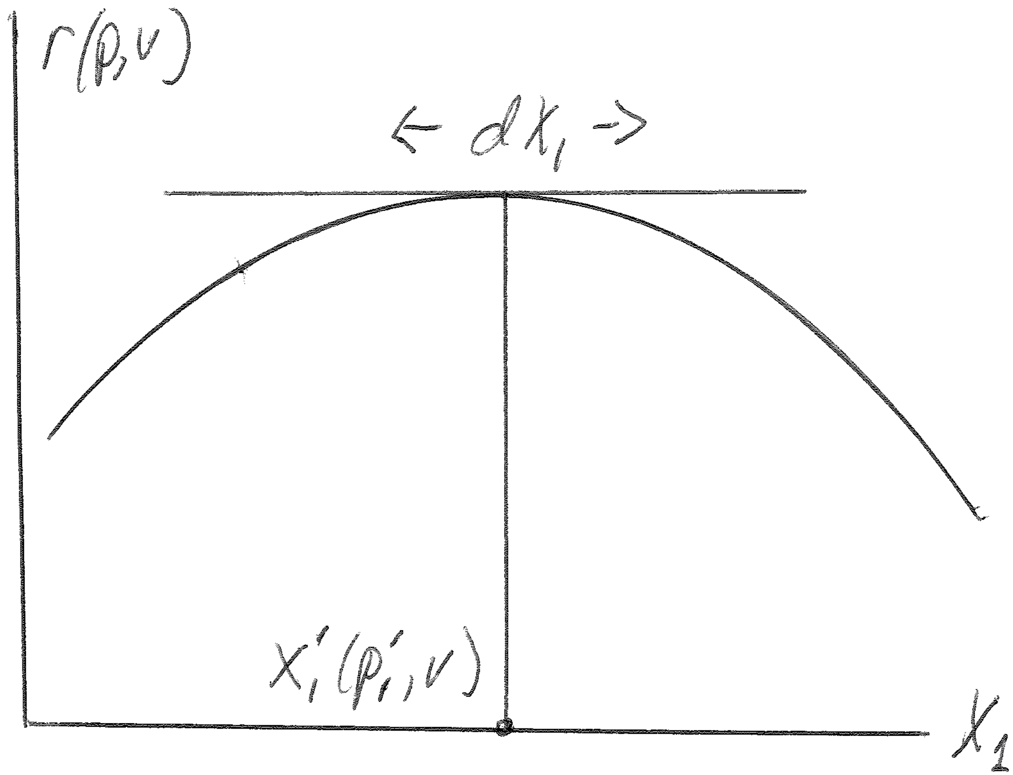
(5) $r_p(p,v) = x$

This is an application of the envelop theorem. Think of r(p,v) as f(p,v) = px(p,v)

$$dr = x_1 dp_1 + \sum_{i=1}^{n} p_i dx_i$$

But if the x's are chosen optimally at the initial p's, then small changes in the X's have no first-order effects on r. Alternatively, the price plane is tangent to the production surface, so

$$\frac{dx_i}{dx_j} = -\frac{p_i}{p_j}$$



(6) $r_{pp}(p,v)p = 0$

Since r(p,v) is homogeneous of degree one in p, its partial derivatives are homogeneous of degree 0. But results from a previous lecture,

$$r_p p = r \qquad r_p + r_{pp} p = r_p \quad \Rightarrow \quad r_{pp} p = 0$$

r_{pp} is positive semi-definite and maps the price vector into zero.

(7) $r_v = w$

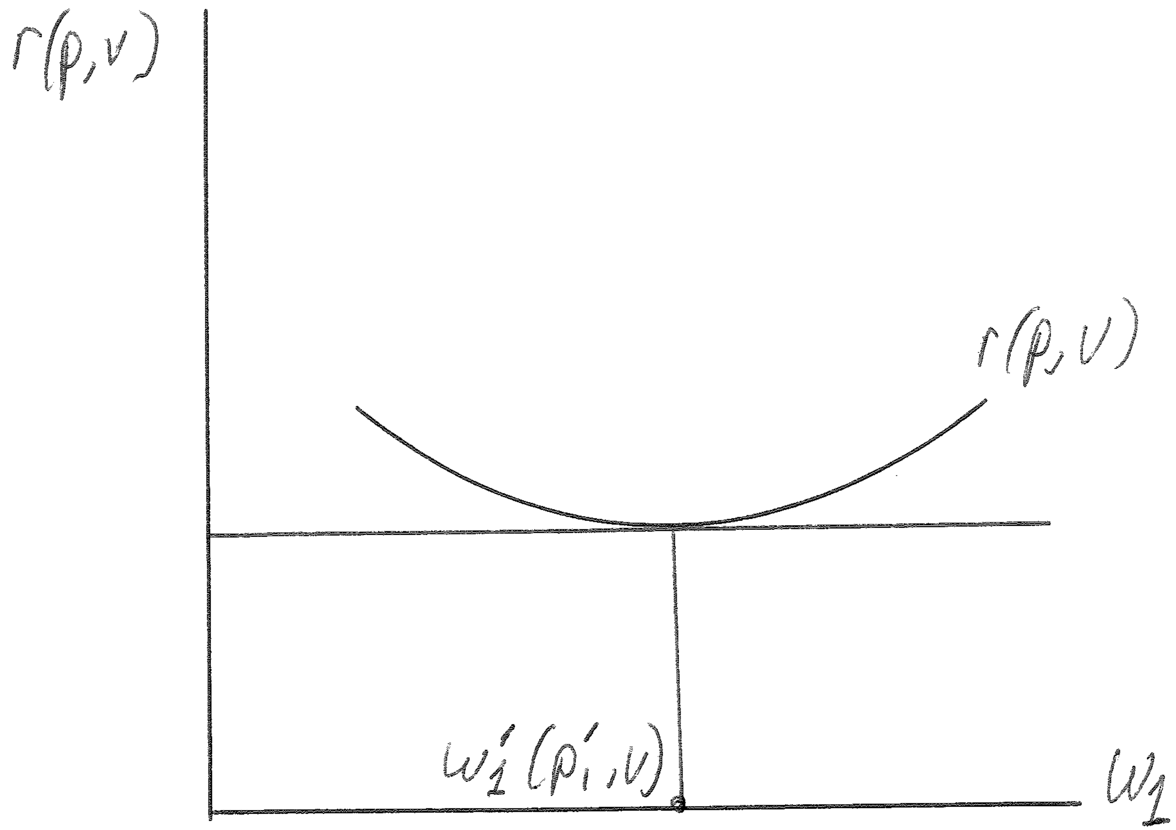
I will not give a proof here (it's in Dixit and Norman), but the revenue function can also be defined as:

$$r(p,v) = \min_w \; w\,v \quad s.t. \quad c(w) \geq p$$

This is again an application of the envelop theorem. Think of r as r(p,v) = w(p,v)v.

$$dr = w_1 dv + \sum_{i=1}^{m} v_i dw_i$$

Buyt if the w's are chosen optimally at the initial p's, then small changes in the w's have no first-order effects on r.



(8) $r_{vv}(p,v)v = 0$

Since r is homogeneous of degree 1 in v, we have from Euler's theorem:

$$r_v v = r$$

differentiate this equation to get

$$r_{vv}v + r_v = r_v \quad => \quad r_{vv}v = 0$$

r_{vv} is negative semi-definite

(9) $$\frac{\partial w_i}{\partial p_j} = r_{v_i p_j} = r_{p_j v_i} = \frac{\partial x_j}{\partial v_i}$$

These are Samuelson's reciprocity relations

These are the basis for the Stolper-Samuelson and Rybczynski theorems. They are duals: if one is true the other must be true.

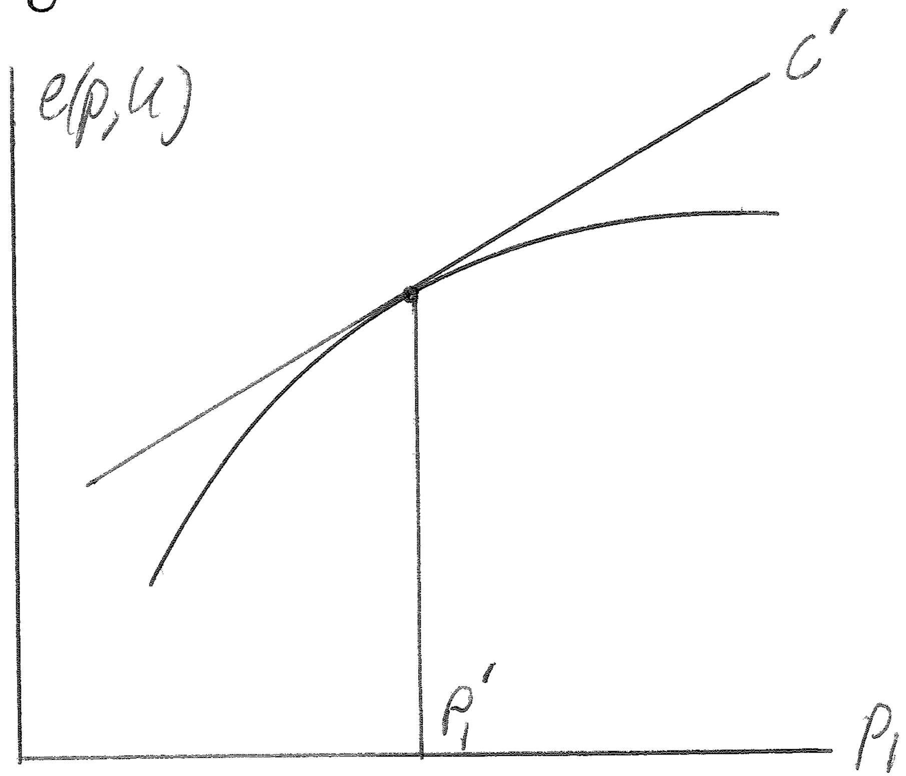
$$(10) \qquad w = \left[r_{vp}\right]p \qquad\qquad x = \left[r_{pv}\right]v$$

$r_v(p,v) = w$ is homogeneous of degree one in p
$r_p(p,v) = x$ is homogeneous of degree one in v

These results then follow using Euler's theorem for homogeneous functions discussed earlier.

Expenditure Function

$$e(p, U) \equiv \min_{c} \; pc \quad s.t. \quad U(c) \geq U$$



(1) homogeneous of degree 1 in p

(2) concave in p

(3) $e_p(p,U) = c$

This is an application of the envelop theorem. Think of e(p,U) as e(p,U) = pc(p,U)

$$de = c_1 dp_1 + \sum_{i=1}^{n} p_i dc_i$$

But if the c's are chosen optimally at the initial p's, then small changes in the c's have no first-order effects on e. Alternatively, the price plane is tangent to the production surface, so

$$\frac{dc_i}{dc_j} = -\frac{p_i}{p_j}$$

(4) $e_{pp}(p,U)p = 0$

$c(p,U) = e_p(p,U)$ is homogeneous of degree zero in p, so this follow from Euler's theorem for homogeneous functions

(5) $e(p,U) = e(p)U$ if U is homogeneous of degree 1

The Excess Demand Function

The price derivatives of e and r and consumption (c) and supply (x) respectively.

Excess Demand for X

$$EDX(p, v) = e_p(p, U) - r_p(p, v)$$

In equilibrium, expenditure equals production,

$$e(p, U) = r(p, v)$$

So we could write

$$U = U(p, v)$$

Therefore

$$EDX(p,v) = e_p(p, U(p,v)) - r_p(p,v)$$

Slope of the excess demand function is given by:

$$\frac{dEDX}{dp} = (e_{pp} - r_{pp}) + e_{pu}\frac{dU}{dp}$$

$$e(p,U) = r(p,v) \qquad \text{in equilibrium}$$

$$e_p dp + e_u dU = r_p dp \qquad \frac{dU}{dp} = \frac{(r_p - e_p)}{e_u} = \frac{-EDX}{e_u}$$

$$\frac{dEDX}{dp} = (e_{pp} - r_{pp}) - EDX\frac{e_{pu}}{e_u}$$

The sign of the income effect depends on the direction of trade. The excess demand function may have a positive slope when X is exported (an increase in price increases income). Note that if e is HD1 in U, and e_p HD1 in U,

$$e_{pu} U = e_p \qquad e_u U = e \qquad \frac{e_{pu}}{e_u} = \frac{e_p}{e} = \frac{c}{e}$$

(marginal propensity to consume c)

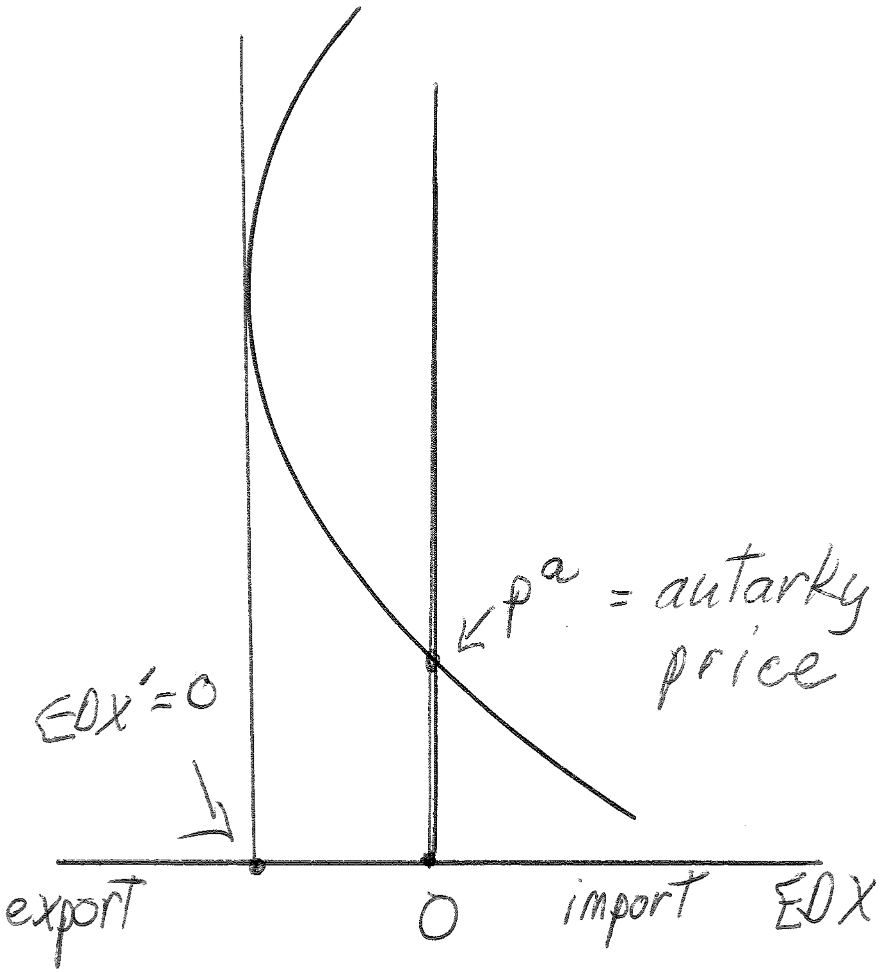
Simpler Version

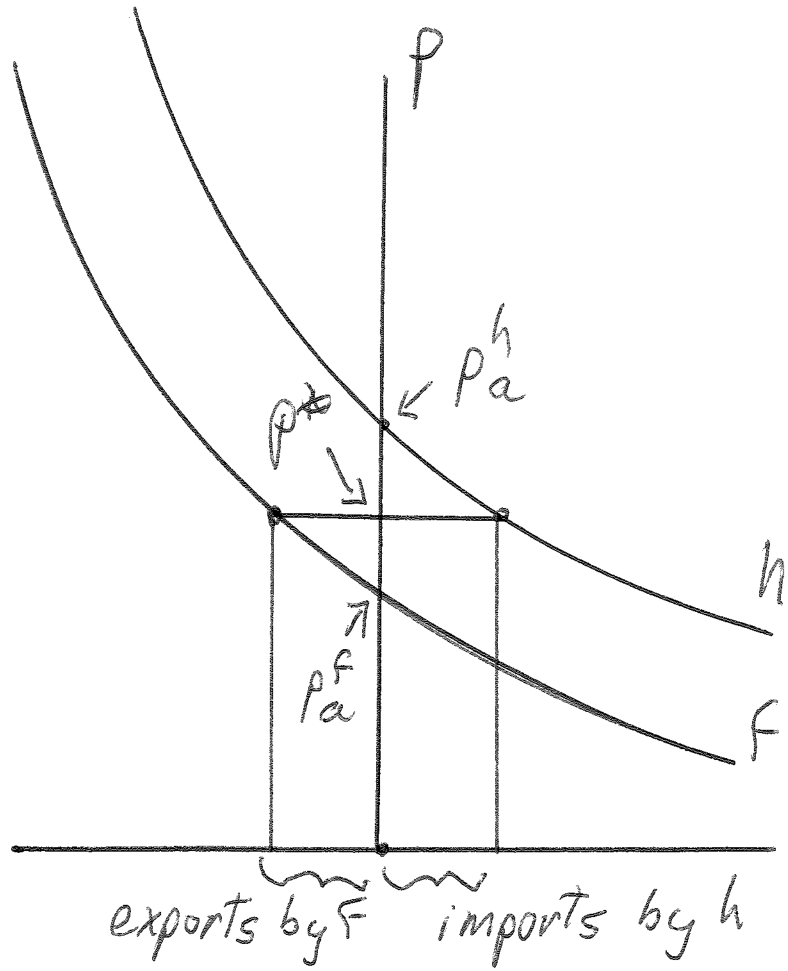
$$EDX = c(p, I(p)) - x(p)$$

$$\frac{d\,EDX}{dp} = \left[\frac{\partial c}{\partial p} - \frac{dx}{dp} \right] + \frac{\partial c}{\partial I} \frac{dI}{dp}$$

The term in brackets is negative.

The change in consumption with respect to income is positive, but the change in income with respect to price is + if X is exported, - if X is imported.





International Equilibrium for countries h and f:

$EDX_h + EDX_f = 0$

Note that if one market clears they must both clear, an application of Walras Law: each "agent" (country) is on its budget constraint, so if one market clears the other one must clear as well.

$pEDX_{h1} = - EDX_{h2}$,

$pEDX_{f1} = - EDX_{f2}$, so

$EDX_{h1} = - EDX_{f1}$ implies $EDX_{h2} = - EDX_{f2}$

Thus one market may be dropped or ignored.

The Offer Function:

Consider two goods X_1 and X_2; how much X_2 is offered in exchange of a given quantity of X_1?

From the excess demand function, we have

$$EDX_i = F_i(p) = e_{pi}(p, U(p, v)) - r_{pi}(p, v)$$

But the balance-of-trade constraint requires that:

$$p_1 EDX_1 + p_2 EDX_2 = 0$$

Only relative prices matter, so set $p_2 = 1$.

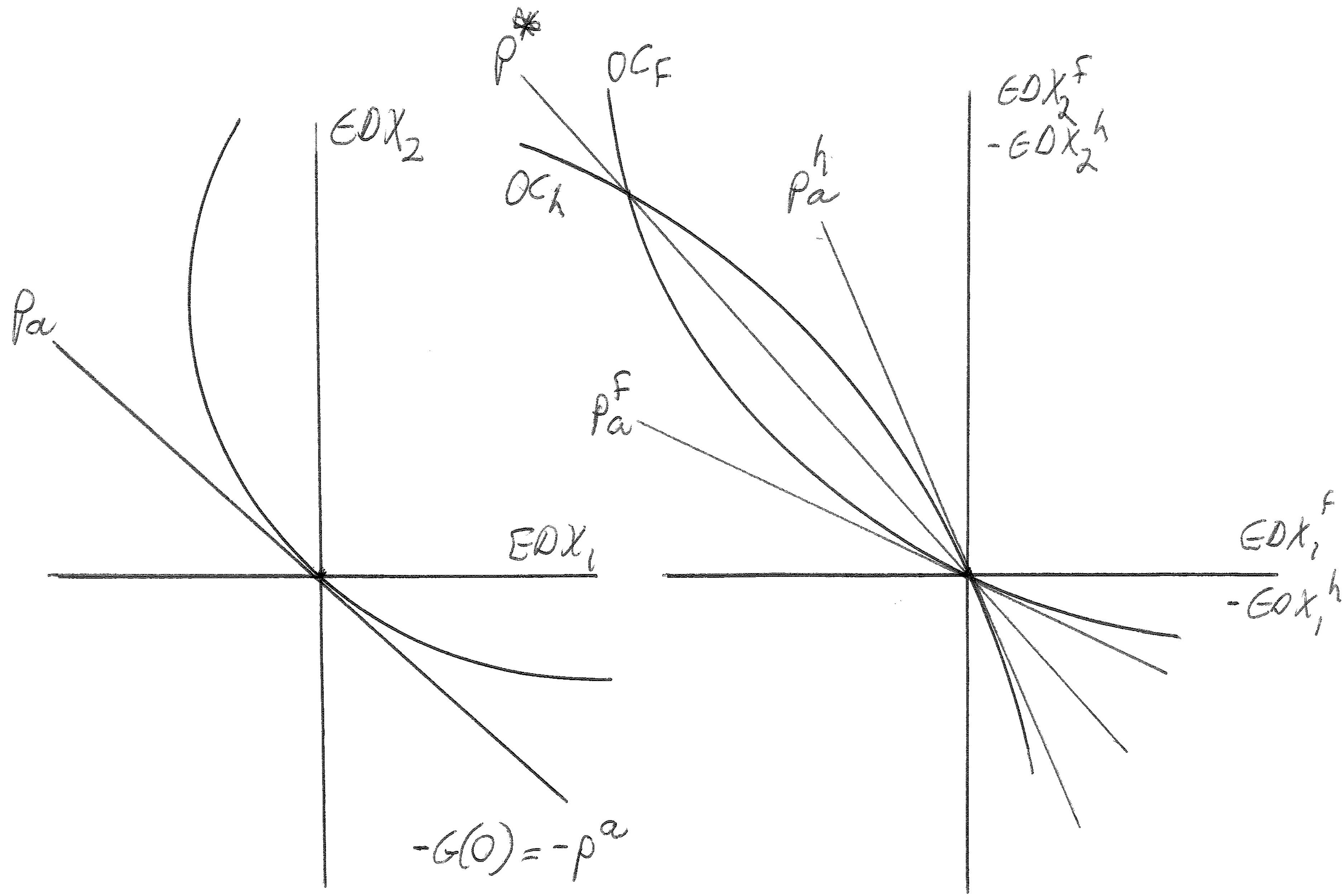
$$EDX_2 = -F^{-1}(EDX_1)(EDX_1) = -G(EDX_1)(EDX_1)$$

$$\frac{dEDX_2}{dEDX_1} = -G'EDX_1 + -G$$

(1) This curve must pass through the origin of EDX1 EDX2 space (balance of trade)

(2) At that origin, the slope is -G(0) which is the autarky price ratio

(3) When the excess demand curve has an infinite slope F' = 0, the offer curve also has an infinite slope G' = inf.



EDX2
Pa
EDX1
-G(0) = -p^a
p*
OCF
OCh
EDX2^f
-EDX2^h
Pa^h
Pa^f
EDX1^f
-EDX1^h