Homework 1

ECON 4818

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Due date: 2.3.2012

1. a) Let X be a random variable with density function given by

$$f(x) = \begin{cases} 2(1-x) & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}.$$

Obtain $E(X)$ and $V(X)$.

b) The joint density of random variables X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} 2 & \text{for } 0 < x_1 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}.$$

b1) Obtain the marginal densities of X_1, X_2. b2) Obtain the conditional density of X_1 given that $X_2 = x_2$ and $E(X_1|X_2 = x_2)$ and $V(X_1|X_2 = x_2)$.

2. Prove the following equalities for any two random variables X and Y.

a) $V(aX + bY) = a^2V(X) + b^2V(Y) + 2abCOV(X, Y)$.
b) Let $\{X_i\}_{i=1}^n$ be a sequence of independent and identically distributed random variables with $E(X_i) = \mu$ and $V(X_i) = \theta$, then $E\left(\sum_{i=1}^n X_i\right) = n\mu$ and $V\left(\sum_{i=1}^n X_i\right) = n\theta$.
c) $V(Y) = E\left(V(Y|X)\right) + V\left(E(Y|X)\right)$.
d) If X and Y are independent then $V(Y|X) = V(Y)$.
e) If $E(Y|X) = \beta_0 + \beta_1 X$ then $\beta_1 = \frac{Cov(X,Y)}{V(X)}$.

2. From your textbook do the following problems:

a) B.4, B.6, B.10 pages 745-746.

1. a) $E(X) = \int_0^1 x \, f(x) \, dx$

$= \int_0^1 x \, 2(1-x) \, dx$

$= \int_0^1 (2x - 2x^2) \, dx$

$= 2 \int_0^1 x \, dx - 2 \int_0^1 x^2 \, dx$

$= 2 \Big|_0^1 \frac{x^2}{2} - 2 \Big|_0^1 \frac{x^3}{3}$

$= 2\left(\frac{1}{2}\right) - 2\left(\frac{1}{3}\right)$

$= 1 - 2/3 = 1/3$

$V(X) = \int_0^1 (x - 1/3)^2 \, 2(1-x) \, dx$

$= 2 \int_0^1 \left(x^2 - \frac{2}{3}x + \frac{1}{9}\right)(1-x) \, dx$

$= 2 \int_0^1 \left(x^2 - 2/3 \, x + 1/9\right) dx$

$- 2 \int_0^1 \left(x^3 - 2/3 \, x^2 + \frac{x}{9}\right) dx$

$= 1/18$

1.b.1) $f_{X_2}(x_2) = \int_0^{x_2} f(x_1, x_2)\, dx_1$

$= \int_0^{x_2} 2\, dx_1$

$= 2 \Big|_0^{x_2} x_1$

$= 2x_2 \qquad 0 < x_2 < 1$

$f_{X_1}(x_1) = \int_{x_1}^{1} 2\, dx_2$

$= 2 \Big|_{x_1}^{1} x_2$

$= 2(1 - x_1) \qquad 0 < x_1 < 1$

1.b.2) $f_{X_1|X_2}(x_1) = \dfrac{f(x_1, x_2)}{f_{X_2}(x_2)}$

$= \dfrac{2}{2x_2} = \dfrac{1}{x_2} \qquad 0 < x_1 < x_2 < 1$

$E(X_1|X_2) = \int_0^{x_2} x_1\, f_{X_1|X_2}(x_1)\, dx_1$

$= \int_0^{x_2} \dfrac{x_1}{x_2}\, dx_1 = \dfrac{1}{x_2} \Big|_0^{x_2} \dfrac{1}{2} x_1^2$

$= \dfrac{1}{x_2} \dfrac{x_2^2}{2} = \dfrac{x_2}{2}, \qquad 0 < x_2 < 1$

$$V(X_1|X_2) = E\left(\left(X_1 - E(X_1|X_2)\right)^2 \Big| X_2\right)$$

$$= E(X_1^2|X_2) - \left(E(X_1|X_2)\right)^2$$

$$= \int_0^{x_2} x_1^2 \frac{1}{x_2}\, dx_1 - \left(\frac{x_2}{2}\right)^2$$

$$= \frac{1}{x_2} \Big|_0^{x_2} \frac{x_1^3}{3} - \frac{x_2^2}{4}$$

$$= \frac{x_2^2}{3} - \frac{x_2^2}{4} = \frac{x_2^2}{12}$$

2. a) $V(aX+bY) = E\left(aX+bY - aE(X) - bE(Y)\right)^2$

$$= E\left(a(X-E(X)) + b(Y-E(Y))\right)^2$$

$$= E\Big(a^2(X-E(X))^2 + b^2(Y-E(Y))^2 + 2ab\,(X-E(X))(Y-E(Y))\Big)$$

$$= a^2V(X) + b^2V(Y) + 2ab\,\mathrm{Cov}(X,Y).$$

b) $E\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n E(X_i)$

$$= \sum_{i=1}^n \mu = n\mu$$

$V\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n V(X_i)$ since X_i, X_j independent

implies $\mathrm{cov}(X_i, X_j) = 0$. So,

$$V\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \theta = n\theta$$

c) $V(Y) = E\left(\left(Y - E(Y)\right)^2\right)$

$= E\left(\left(Y - E_X\left(E(Y|X)\right)\right)^2\right)$

$= E\left(Y^2 - 2Y\,E_X\left(E(Y|X)\right) + \left\{E_X\left(E(Y|X)\right)\right\}^2\right)$

$= E(Y^2) - 2\,E(Y)E_X\left(E(Y|X)\right) + \left(E_X\left(E(Y|X)\right)\right)^2$.

$= E_X\left(E(Y^2|X)\right) - 2\,E_X\left(E(Y|X)\right)E_X\left(E(Y|X)\right)$

$+ \left(E_X\left(E(Y|X)\right)\right)^2$

$= E_X\left(E(Y^2|X)\right) - \left\{E_X\left(E(Y|X)\right)\right\}^2$

$= E_X\left(E(Y^2|X) - \left(E(Y|X)\right)^2 + \left(E(Y|X)\right)^2\right)$

$- \left\{E_X\left(E(Y|X)\right)\right\}^2$

$= E_X\left(V(Y|X) + \left(E(Y|X)\right)^2\right)$

$- \left\{E_X\left(E(Y|X)\right)\right\}^2$

$E_X\left(V(Y|X)\right) + E_X\left(E(Y|X)^2\right) - \left\{E_X\left(E(Y|X)\right)\right\}^2$

$= E_X\left(V(Y|X)\right) + V_X\left(E(Y|X)\right)$

d) $V(Y|X) = E(Y^2|X) - \left\{E(Y|X)\right\}^2$, but by independence $E(Y|X) = E(Y)$ and $E(Y^2|X) = E(Y^2)$

Hence, $V(Y|X) = V(Y)$.

e) $E(E(Y|X)) = \beta_0 + \beta_1 E(X)$ and by LIE

$$E(Y) = \beta_0 + \beta_1 E(X)$$

$$\begin{aligned} Y - E(Y) &= Y - \beta_0 - \beta_1 E(X) \\ &= \cancel{\beta_0} + \beta_1 X + U - \cancel{\beta_0} - \beta_1 E(X) \\ &= \beta_1 (X - E(X)) + U, \quad U = Y - E(Y|X). \end{aligned}$$

Then,

$$(X - E(X))(Y - E(Y)) = \beta_1 (X - E(X))^2 + U(X - E(X))$$

and

$$E(X - E(X))(Y - E(Y)) = \beta_1 E(X - E(X))^2 + E(U(X - E(X)))$$

or

$$Cov(X,Y) = \beta_1 V(X) + E(U(X - E(X)))$$

But
$$\begin{aligned} E(U(X - E(X))) &= E_X(E(U(X - E(X)) \mid X)) \\ &= E_X((X - E(X))\, E(U|X)) \\ &= 0 \quad \text{since } E(U|X) = 0 \end{aligned}$$

So, $Cov(X,Y) = \beta_1 V(X) \Rightarrow \beta_1 = \dfrac{Cov(X,Y)}{V(X)}$.

3. B.4.
$$\begin{aligned} P(X \geq .6) &= 1 - P(X < .6) \\ &= 1 - F(.6) \\ &= 1 - (3 \times (.6)^2 - 2(.6)^3). \end{aligned}$$

B.6. $\int x f(x)\, dx = \int_0^3 \frac{1}{9} x^3\, dx = \frac{1}{9} \Big|_0^3 \frac{x^4}{4}$

$\frac{1}{9} \frac{3^4}{4} = 2.25$

B 10.

(i) $E(GPA \mid SAT = 800) = .7 + 0.002 . 800 = 2.3$

$E(GPA \mid SAT = 1400) = .7 + 0.002 . 1400 = 3.5$

Difference : $3.5 - 2.3 = 1.2$

A student with SAT = 1400 is expected to have a GPA that is 1.2 points higher than a student with SAT = 800

(ii) $E\big(E(GPA \mid SAT)\big) = .7 + 0.002\, E(SAT)$

$= .7 + 0.002 \times 1100$

$= 2.9$

(iii) No. It only means that his/her GPA is expected to be 2.9