Some solutions

3.5.1 (a) true, (b) true, (c) false, \([0, 1]\) is uncountably infinite yet compact, (d) true by Lemma 3.5.4 and Theorem 3.5.5, (e) False, consider \(S = [0, 1) \cup (2, 3]\) which has a max and a min yet is not compact by Theorem 3.5.5.

3.5.3 (a) Consider \(F = \{U_n : n \in \mathbb{N}\}\) where \(U_n = (\frac{1}{2^n}, 3 - \frac{1}{n})\). The sets \(U_n\) are open intervals that are getting bigger and bigger and the union of them all covers \([1, 3)\) but no finite subcollection will cover the right hand side of the interval.

3.5.3 (c) Consider \(F = \{W_n : n \in \mathbb{N}\}\) where \(W_n = (n - \frac{1}{2}, n + \frac{1}{n})\). The \(W_n\) are open intervals each having length 1 and we have \(W_n \cap \mathbb{N} = \{n\}\) for all \(n \in \mathbb{N}\). Hence if we take the union of only finitely many \(W_n\) we cover only finitely many positive integers, and not all of them.

3.5.6 (a) If we let \(A_n = [n, \infty)\) for each \(n \in \mathbb{N}\), then \(A_n\) is closed (but not bounded) for all \(n \in \mathbb{N}\). We have \(A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq A_{n+1} \supseteq \cdots\) and \(\cap_{n=1}^{\infty} A_n = \emptyset\).

3.5.6 (b) If we let \(A_n = (0, \frac{1}{n})\) for each \(n \in \mathbb{N}\), then \(A_n\) is bounded (but not closed) for all \(n \in \mathbb{N}\). We have \(A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq A_{n+1} \supseteq \cdots\) and \(\cap_{n=1}^{\infty} A_n = \emptyset\).

4.1.2 (b) False- let \(s_n = -n\). Then \(s_n < \epsilon\) for all \(n\) but \((s_n)\) diverges to \(-\infty\).

4.1.7 (a) Let \(\epsilon > 0\). Note that

\[
|\frac{1}{2 + 3n} - 0| = |\frac{1}{2 + 3n}| < \frac{1}{3n} = \frac{1}{3n}.
\]

We note that

\[
\frac{1}{3n} < \epsilon
\]

whenever

\[
n > \frac{1}{3\epsilon}.
\]

So choose \(N > \frac{1}{3\epsilon}\). Then if \(n \geq N\), \(\frac{1}{3n} < \epsilon\), so that

\[
|\frac{1}{2 + 3n} - 0| = \frac{1}{2 + 3n} < \frac{1}{3n} < \epsilon.
\]

It follows from the definition that

\[
\lim_{n \to \infty} \frac{1}{2 + 3n} = 0.
\]
4.1.7 (c) Let \( \epsilon > 0 \). Note that
\[
\left| \frac{6n^2 + 3n}{2n^2 - 5} - 3 \right| = \left| \frac{6n^2 + 3n - 3(2n^2 - 5)}{2n^2 - 5} \right| = \left| 3n + 15 \right|.
\]
For all \( n \), we know that \( |3n + 15| = 3n + 15 \leq 3n + 15n = 18n \). And for all \( n \geq 2 \),
\[
2n^2 - 5 \geq 2n^2 - (3n^2)/2 = (n^2)/2 > 0.
\]
Thus
\[
\frac{1}{2n^2 - 5} \leq \frac{2}{n^2}
\]
for all \( n \geq 2 \). It follows that for all \( n \geq 2 \),
\[
\left| \frac{3n + 15}{2n^2 - 5} \right| \leq 18n \cdot \frac{2}{n^2} = \frac{36}{n}.
\]
Thus if \( \frac{36}{n} < \epsilon \) we will have
\[
\left| \frac{6n^2 + 3n}{2n^2 - 5} - 3 \right| \leq \frac{36}{n} < \epsilon.
\]
Choose \( N > \frac{36}{\epsilon} \). Then if \( n \geq N \), \( \frac{36}{n} < \epsilon \). So for all \( n > N \),
\[
\left| \frac{6n^2 + 3n}{2n^2 - 5} - 3 \right| < \epsilon.
\]
It follows from the definition that
\[
\lim_{n \to \infty} \frac{6n^2 + 3n}{2n^2 - 5} = 3.
\]

4.1.9 (a) True. Suppose \( \lim_{n \to \infty} s_n = s \). Then fixing \( \epsilon > 0 \), there exists \( N > 0 \) such that for all \( n > N \),
\[
|s_n - s| < \epsilon.
\]
But by problem 11.6 (a),
\[
||s_n| - |s|| \leq |s_n - s|.
\]
It follows that for all \( n > N \),
\[
||s_n| - |s|| < \epsilon.
\]
Hence
\[
\lim_{n \to \infty} |s_n| = |s|.
\]

4.1.9 (b) False. Consider \( (s_n) = ((-1)^n) \). We have \( (|s_n|) = (|((-1)^n)|) = (1) \) which is convergent, since it is the constant sequence equal to 1 for all \( n \in \mathbb{N} \). But, on the other hand, \( (s_n) = (-1, 1, -1, 1, \ldots) \) is divergent. It is the alternating series that alternates between \(-1\) and 1.