4.4.3 (a) We have done this in class and shown $S = \{-1, 1\}$ so that $\lim \inf s_n = -1$ and $\lim \sup s_n = 1$.

4.4.3 (b) The $(2k-1)^{st}$ and $(2k)^{th}$ terms of this sequence are given by $\frac{1}{2k}$ and $\frac{1}{2k-1}$ respectively. Thus this sequence converges to 0. In this case $S = \{0\}$ and

$$\lim \inf s_n = \lim \sup s_n = \lim_{n \to \infty} s_n = 0.$$  

4.4.3 (c) For $n$ odd, $u_n = n^2(-1 + 1) = -2n^2$. For $n$ even, $u_n = n^2(-1 + 1) = n^2 \cdot 0 = 0$. Thus $\lim \inf s_n = -\infty$ and $\lim \sup s_n = 0$.

4.4.3 (d) We first consider the patterns in $\sin \frac{n\pi}{2}$. For $n = 1 \mod 4$, $\sin \frac{n\pi}{2} = \sin \frac{\pi}{2} = 1$. For $n = 2 \mod 4$, $\sin \frac{n\pi}{2} = \sin \frac{2\pi}{2} = 0$. For $n = 3 \mod 4$, $\sin \frac{n\pi}{2} = \sin \frac{3\pi}{2} = -1$. For $n = 4 \mod 4$, $\sin \frac{n\pi}{2} = \sin \frac{4\pi}{2} = 0$. Thus the sequence $(\sin \frac{n\pi}{2})$ cycles through a pattern of 4, like so: $(1, 0, -1, 0, 1, 0, -1, 0, \cdots)$

Hence

$$(n \sin \frac{n\pi}{2}) = (1, 0, -3, 0, 5, 0, -7, 0, 9, 0, -11, 0, \cdots)$$

and the extended set of subsequential limits is equal to $\{-\infty, 0, +\infty\}$. Thus $\lim \inf s_n = -\infty$ and $\lim \sup s_n = +\infty$.

4.4.7 (a) TRUE: By Theorem 4.4.7, every bounded sequence has a convergent subsequence $(s_{n_k})$, and since $(s_{n_k})$ is convergent, it is a Cauchy sequence.

4.4.7 (b) FALSE: if the monotone sequence is unbounded, any subsequence is also unbounded. For example if we take $(s_n) = (n^2)$ this sequence is increasing, hence monotone, and unbounded, and all of its subsequences also are unbounded and tend to $+\infty$.

4.4.7 (c) TRUE: Suppose $(s_n)$ converges with $\lim s_n = s \in \mathbb{R}$. Let $t_n = s_n + (-1)^n$, and let $u_n = (-1)^{n+1}$. One checks that $t_n + u_n = s_n + (-1)^n + (-1)^{n+1} = s_n + 0 = s_n$. Thus $(s_n)$ is the sum of $(t_n)$ and $(u_n)$. We calculate that $\lim \inf t_n = s - 1$ and $\lim \sup t_n = s + 1$, so that $(t_n)$ is oscillating. Also $\lim \inf u_n = -1$ and $\lim \sup u_n = 1$, so that $(u_n)$ is oscillating.

4.4.11 Let $S$ denote the set of subsequential limits of the set $(s_n)$. To show that $S$ is closed, it suffices to show that $S'$, the set of accumulation points of $S$, is contained in $S$. Let $x \in S'$. Put $\epsilon = \frac{1}{2}$, find $x_1 \in N^*(x; \frac{1}{2}) \cap S$, which is nonempty since $x$ is an accumulation point of $S$. Then by definition of $S$, $x_1$ is a subsequential limit of $(s_n)$, so we can find $n_1$ with $|s_{n_1} - x_1| < \frac{1}{2}$. Then using the triangle inequality, we see
that

\[ |s_{n_1} - x| = |s_{n_1} - x_1 + x_1 - x| \leq |s_{n_1} - x_1| + |x_1 - x| < \frac{1}{2} + \frac{1}{2} = 1. \]

Continue on in this way, and suppose we have constructed positive integers
\( n_1 < n_2 < \cdots < n_k \) such that

\[ |s_{n_i} - x| < \frac{1}{k}, \quad 1 \leq i \leq k. \]

At the \((k + 1)\)st stage, we select \( x_{k+1} \in N^*(x; \frac{1}{2(k+1)}) \cap S \), then noting that \( x_{k+1} \) is a subsequential limit of \((s_n)\), we can find \( n_{k+1} \) with \( n_{k+1} > n_k \) and

\[ |s_{n_{k+1}} - x_{k+1}| < \frac{1}{2(k + 1)}. \]

Using the triangle inequality again, we see that

\[ |s_{n_{k+1}} - x| = |s_{n_{k+1}} - x_{k+1} + x_{k+1} - x| \leq |s_{n_{k+1}} - x_{k+1}| + |x_{k+1} - x| < \frac{1}{2(k + 1)} + \frac{1}{2(k + 1)} = \frac{1}{k + 1}. \]

It is now evident that the subsequence \((s_{n_k})\) converges to \( x \), by Theorem 16.14, since \( |s_{n_k} - x| < \frac{1}{k} \) for all \( k \in \mathbb{N} \) and \( \lim_{k \to \infty} \frac{1}{k} = 0 \). Thus we have proven \( x \in S \), so that \( S' \subset S \), and \( S \) is closed.

5.1.3(a) By Theorem 5.1.13,

\[
\lim_{x \to 1} \frac{3x^2 + 5}{x^3 + 1} = \frac{3 \cdot (-1)^2 + 5}{1 + 1} = \frac{8}{2} = 4.
\]

5.1.3(c) We note that

\[
\frac{\sqrt{x} - 1}{x - 1} = \frac{\sqrt{x} - 1}{(\sqrt{x} + 1)(\sqrt{x} + 1)} = \frac{1}{\sqrt{x} + 1}
\]

so that

\[
\lim_{x \to 1} \sqrt{x} - 1 x - 1 = \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{1 + 1} = \frac{1}{2},
\]

with the last equalities coming from Theorem 5.1.13.

5.1.3(h) We first note that for \( x > 1 \), \( x - 1 > 0 \), so that \( |x - 1| = x - 1 \) whenever \( x > 1 \). Therefore:

\[
\lim_{x \to 1+} \frac{x^2 - 1}{|x - 1|} = \lim_{x \to 1+} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1+} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \to 1+} (x + 1) = 2.
\]

5.1.9(c) Note for \( x > 0 \) we have \( -1 \leq \sin \frac{1}{x} \leq 1 \) so that

\[-x \leq x \sin \frac{1}{x} \leq x, \quad x > 0.\]
Thus given \( \epsilon > 0 \), take \( \delta = \epsilon \), and then whenever \( 0 < x < \epsilon = \delta \),
\[
|x \sin \frac{1}{x} - 0| = |x \sin \frac{1}{x}| \leq |x| < \epsilon.
\]

Thus
\[
\lim_{x \to 0^+} x \sin \frac{1}{x} = 0.
\]

5.1.4 This problem is incorrect as stated. It’s clear that as \( x \to 2 \), \( |x^2 + 2x - 18| \to |10 - 10| = |10| \). However if we change the problem slightly and try to find \( \delta \) such that whenever \( 0 < |x - 2| < \delta \) we have \( |x^2 + 2x - 8| < \frac{1}{4} \), note that
\[
x^2 + 2x - 8 = (x + 4)(x - 2),
\]
and if \( 0 < |x - 2| < 1 \) we have \( 6 < x + 4 < 7 \), so that if \( 0 < |x - 2| < 1 \), \( |x + 4| < 7 \). Therefore for \( 0 < |x - 2| < 1 \),
\[
|x^2 + 2x - 8| = |(x + 4)(x - 2)| = |x + 4||x - 2| < 7|x - 2|.
\]

It follows that if we take \( 7|x - 2| < \frac{1}{4} \), i.e. \( |x - 2| < \frac{1}{28} \), we have also that \( |x - 2| < 1 \) and these two inequalities give:
\[
|x^2 + 2x - 8| = |x + 4||x - 2| < 7|x - 2| < 7 \cdot \frac{1}{28} < \frac{1}{4}.
\]

(There are other possible answers too!!).

5.1.16 Let \( \lim_{x \to c} f(x) = L > 0 \). Now take \( \epsilon = L \) in the definition of limit. Then there exists \( \delta > 0 \) such that
\[
|f(x) - L| < L \text{ whenever } 0 < |x - c| < \delta \text{ and } x \in D.
\]

Let \( U = N^*(c; \delta) \). Then if \( x \in U \cap D \),
\[
-L < f(x) - L < L
\]
so that adding \( L \) to all sides of the inequality we obtain
\[
0 < f(x) < L + L = 2L
\]
whenever \( x \in U \cap D \). Thus \( f(x) > 0 \) for all \( x \in D \cap U \).

5.1.18 Let \( \lim_{x \to c} f(x) = L \). Take \( \epsilon = 1 \) in the definition of limit. Then there exists \( \delta > 0 \) such that
\[
|f(x) - L| < 1 \text{ whenever } 0 < |x - c| < \delta \text{ and } x \in D.
\]

Since the reverse triangle inequality shows us \( |f(x)| - |L| \leq |f(x) - L| \), we obtain
\[
|f(x)| - |L| < 1 \text{ whenever } 0 < |x - c| < \delta \text{ and } x \in D.
\]
and adding $L$ to both sides gives

$$|f(x)| < |L| + 1 \text{ whenever } 0 < |x - c| < \delta \text{ and } x \in D.$$ 

Let $M = \max\{|L| + 1, |f(c)|\}$, and let $U = N(c; \delta)$. Then we have

$$|f(x)| \leq M \text{ whenever } x \in U \cap D,$$

so that $f$ is bounded on a neighborhood of $D$. 