Q. 1(a) Consider a Cauchy sequence of sequences in \( \mathbb{C} \):

\[ \{(c_k^{(n)})_{k=1}^{\infty}\}_{n=1}^{\infty}, \]

where

\[ \lim_{k \to \infty} c_k^{(n)} = c_0^{(n)} \in \mathbb{C}. \]

Since the sequence is Cauchy in the given norm on \( \mathbb{C} \), we have that for every \( \varepsilon > 0 \), there exists \( N > 0 \) such that whenever \( n > m \geq N \),

\[ \| (c_k^{(n)})_{k=1}^{\infty} - (c_k^{(m)})_{k=1}^{\infty} \| < \varepsilon/2, \]

i.e. whenever \( n > m \geq N \),

\[ \sup_{k \in \mathbb{N}} |c_k^{(n)} - c_k^{(m)}| < \varepsilon/2. \]

But this means that fixing \( k_0 \in \mathbb{N} \), there exists \( N > 0 \) such that whenever \( n > m \geq N \),

\[ |c_k^{(n)} - c_k^{(m)}| < \varepsilon/3. \]

So for fixed \( k_0 \in \mathbb{N} \), the sequence \( \{c_k^{(n)}\}_{n=1}^{\infty} \), is a Cauchy sequence, and hence converges to some complex number \( c_{k_0}^{(0)} \), i.e.

\[ \lim_{n \to \infty} c_k^{(n)} = c_{k_0}^{(0)}. \]

We now claim that the sequence

\[ (c_{k_0}^{(0)})_{k=1}^{\infty} \]

is Cauchy, hence converges to some complex number \( L \), and moreover, that

\[ \lim_{n \to \infty} \| (c_k^{(n)})_{k=1}^{\infty} - (c_{k_0}^{(0)})_{k=1}^{\infty} \| = 0 \]

so that \( (c_{k_0}^{(0)})_{k=1}^{\infty} \) is the limit of \( \{(c_k^{(n)})_{k=1}^{\infty}\}_{n=1}^{\infty} \) in \( \mathbb{C} \).

We now recall that for our fixed \( \varepsilon > 0 \), we have found \( N > 0 \) such that whenever \( n > m \geq N \),

\[ \sup_{k \in \mathbb{N}} |c_k^{(n)} - c_k^{(m)}| < \varepsilon/3. \]

Thus if \( n > m \geq N \), and \( k \in \mathbb{N} \) is fixed, and \( k \in \mathbb{N} \) is arbitrary,

\[ |c_k^{(n)} - c_k^{(m)}| < \varepsilon/3. \]
Fixing $m \geq N$, and letting $n \to \infty$, we get that whenever $m \geq N$, and $k \in \mathbb{N}$ is arbitrary,

$$\lim_{n \to \infty} |c_k^{(n)} - c_k^{(m)}| = |c_k^{(0)} - c_k^{(m)}| \leq \varepsilon/3.$$  

Thus, whenever $m \geq N$,

$$\sup_{k \in \mathbb{N}} |c_k^{(0)} - c_k^{(m)}| \leq \varepsilon/3 < \varepsilon.$$  

We now show that $(c_k^{(0)})$ is a Cauchy sequence. Choose $m_0 \geq N$. Then by hypothesis, $(c_k^{(m_0)})_{k=1}^{\infty}$ is a Cauchy sequence of real numbers, so that there exists $N' \in \mathbb{N}$ such that whenever $i > j \geq N'$,

$$|c_i^{(m_0)} - c_j^{(m_0)}| < \varepsilon/3.$$  

Since whenever $m \geq N$ we have that

$$\sup_{k \in \mathbb{N}} |c_k^{(0)} - c_k^{(m)}| \leq \varepsilon/3,$$  

and since $m_0 \geq N$, we have for every $k \in \mathbb{N}$,

$$|c_k^{(0)} - c_k^{(m_0)}| \leq \varepsilon/3.$$  

It follows that for $i > j \geq N'$,

$$|c_i^{(0)} - c_j^{(0)}| = |c_i^{(0)} - c_i^{(m_0)} + c_i^{(m_0)} - c_j^{(m_0)} + c_j^{(m_0)} - c_j^{(0)}| \leq |c_i^{(0)} - c_i^{(m_0)}| + |c_i^{(m_0)} - c_j^{(m_0)}| + |c_j^{(m_0)} - c_j^{(0)}|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$  

We then see that $(c_k^{(0)})_{k=1}^{\infty}$ is a Cauchy sequence, hence belongs to $\mathbf{c}$. Since we have already shown that whenever $m \geq N$, 

$$\sup_{k \in \mathbb{N}} |c_k^{(0)} - c_k^{(m)}| < \varepsilon,$$  

we know that whenever $m \geq N$, 

$$\|(c^{(0)})_{k=1}^{\infty} - (c^{(m)})_{k=1}^{\infty}\| < \varepsilon,$$  

and

$$\lim_{m \to \infty} (c^{(m)})_{k=1}^{\infty} = (c^{(0)})_{k=1}^{\infty} \in \mathbf{c}.$$  

p. 155, Q. 4 Suppose $\{T_n\}_{n=1}^{\infty} \subset \mathcal{L}(\mathcal{X}, \mathcal{Y})$ with $T_n \to T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, and $\{x_n\}_{n=1}^{\infty} \subset \mathcal{X}$ with $x_n \to x \in \mathcal{X}$. We have shown in lectures that $\|T_n\| \to \|T\| \in [0, \infty)$. Let $\varepsilon = 1 > 0$ in the definition of convergence. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, 

$$\|T_n - T\| < \varepsilon.$$  

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By the reverse triangle inequality for norms, we have
\[ \|T_n\| - \|T\| < \|T_n - T\| < 1, \quad \forall n \geq N. \]
Therefore
\[ \|T_n\| < 1 + \|T\|, \quad \forall n \geq N. \]
Let \( M = \max\{\|T_1\|, \|T_2\|, \ldots, \|T_{N-1}\|, 1 + \|T\|\} \). Then,
\[ \|T_n\| \leq M, \quad \forall n \in \mathbb{N}, \]
i.e. the \( \{T_n\} \) are uniformly bounded in operator norm. We now compute that:
\[ \|T_n(x_n) - T(x)\| = \|T_n(x_n) - T_n(x) + T_n(x) - T(x)\| \leq \|T_n(x_n - x)\| + \|T_n - T\| \cdot \|x\| \]
\[ \leq \|T_n\| \cdot \|x_n - x\| + \|T_n - T\| \cdot \|x\| \leq M \cdot \|x_n - x\| + \|T_n - T\| \cdot \|x\|. \]
Now \( M \) and \( \|x\| \) are non-negative constants and \( \|x_n - x\| \) and \( \|T_n - T\| \) go to 0 as \( n \to \infty \). It follows that
\[ \lim_{n \to \infty} \|T_n(x_n) - T(x)\| = 0 \]
so that \( T_n(x_n) \to T(x) \) in \( \mathcal{Y} \).

p. 164, Q. 27 The rational numbers \( \mathbb{Q} \) are countable, so we can enumerate them like so:
\[ \mathbb{Q} = \{q_i\}_{i=1}^\infty. \]
For each \( n \in \mathbb{N} \), and each \( i \in \mathbb{N} \), let \( B(\frac{1}{n \cdot 2^i}, q_i) \) be the open interval of \( \mathbb{R} \) centered at \( q_i \) having radius \( \frac{1}{n \cdot 2^i} \). We note that taking \( m \) to be Lebesgue measure on \( \mathbb{R} \), we have
\[ m(B(\frac{1}{n \cdot 2^i}, q_i)) = 2 \cdot \frac{1}{n \cdot 2^i} = \frac{1}{n \cdot 2^{i-1}}. \]
Fixing \( n \in \mathbb{N} \), let \( U_n = \bigcup_{i=1}^\infty B(\frac{1}{n \cdot 2^i}, q_i) \). Each \( U_n \) is open, since it is a union of open balls, and each \( U_n \) is dense in \( \mathbb{R} \), since each \( U_n \) contains the dense set \( \mathbb{Q} \). Also by countable subadditivity of Lebesgue measure we have:
\[ m(U_n) = m(\bigcup_{i=1}^\infty B(\frac{1}{n \cdot 2^i}, q_i)) \leq \sum_{i=1}^\infty m(B(\frac{1}{n \cdot 2^i}, q_i)) \]
\[ = \sum_{i=1}^\infty \frac{1}{n \cdot 2^{i-1}} = \frac{2}{n}. \]
We note that setting \( U = \bigcap_{n=1}^\infty U_n \), we have since the \( U_n \) are nested decreasing that
\[ m(U) = \lim_{n \to \infty} \frac{2}{n} = 0. \]
Now \( U \) is has measure 0, and it is the complement of a meager set, since taking \( A_n = \mathbb{R} \setminus U_n \), each \( A_n \) is closed and nowhere dense since \( U_n \) is open and dense. By DeMorgan’s laws, setting \( A = \bigcup_{n=1}^\infty A_n \), we know that \( A \) is a meager set, i.e. a set of the first category, and its complement is \( U \) which has measure 0.
From the definition on p. 162-163 of the Folland textbook, we see that an operator
\[ T : \mathcal{X} \to \mathcal{Y} \]
is closed if its graph,
\[ \Gamma(T) = \{(x, T(x)) : x \in \mathcal{X}\} \subset \mathcal{X} \times \mathcal{Y} \]
is closed as a subspace of \( \mathcal{X} \times \mathcal{Y} \). So, we suppose \{f_i\}_{i=1}^{\infty} \subset \mathcal{X} \) is given and that \{(f_i, T(f_i))\}_{i=1}^{\infty} \) converges to \((f, g) \in \mathcal{X} \times \mathcal{Y} \). This means \{f_i\}_{i=1}^{\infty} \) converges to \( f \) in \( L^1 \) norm and that \{T(f_i)\}_{i=1}^{\infty} \) converges to \( g \) in \( L^1 \) norm. We want to prove that this implies \( T(f) = g \). We know that by the definition of norm in \( L^1(\mu) \),
\[ \lim_{i \to \infty} \sum_{n=1}^{\infty} |f_i(n) - f(n)| = 0. \]
Since for each fixed \( n_0 \in \mathbb{N} \) we have \(|f_i(n_0) - f(n_0)| \leq \sum_{n=1}^{\infty} |f_i(n) - f(n)|\), this gives
\[ \lim_{i \to \infty} |f_i(n_0) - f(n_0)| = 0 \]
so that \( \lim_{i \to \infty} f_i(n_0) = f(n_0) \), \( \forall \ n_0 \in \mathbb{N} \). Similarly,
\[ \lim_{i \to \infty} \sum_{n=1}^{\infty} |T(f_i)(n) - g(n)| = \lim_{i \to \infty} \sum_{n=1}^{\infty} |nf_i(n) - g(n)| = 0. \]
By the same argument we see that for each fixed \( n_0 \in \mathbb{N} \) we have
\[ \lim_{i \to \infty} n_0 f_i(n_0) = g(n_0). \]
But,
\[ \lim_{i \to \infty} n_0 f_i(n_0) = n_0 \cdot f(n_0), \ \forall n_0 \in \mathbb{N}. \]
Since \( g \in L^1(\mu) = \mathcal{Y} \), and \( g(n) = nf(n) \) \( \forall n \in \mathbb{N} \), we get that \( f \in \mathcal{X} \) with \( T(f) = g \). It follows that \( (f, g) = (f, T(f)) \in \Gamma(T) \), so that the graph of \( T \), \( \Gamma(T) \) is closed, and therefore \( T \) is closed.