University of Colorado

Department of Mathematics

2016/2017 Semester 2 | Math 8340 Functional Analysis 2 | Assignment 2

Due Monday Feb. 27, 2017

1. Do Exercises II.4.3, II.4.8, II.4.11, II.4.12 on pp. 45–46 of the Conway textbook.

2. Let $\mathcal{H}_1 = \ell^2(\mathbb{Z})$, and let $\mathcal{H}_2 = L^2[0,1]$. Show that the map $\mathcal{F} : \mathcal{H}_1 \to \mathcal{H}_2$ defined by $\mathcal{F}((a_j)) = \sum a_j e^{2\pi ijt}$ is an isomorphism of Hilbert spaces. Let U denotes the bilateral shift operator on $\mathcal{H}_1$ defined by

$$U(e_n) = e_{n-1}, \ n \in \mathbb{Z},$$

where $\{e_n\}_{n\in\mathbb{Z}}$ is the standard orthonormal basis for $\ell^2(\mathbb{Z})$, so that $e_n(j) = 1$ if $n = j$ and 0 otherwise. Prove that $\mathcal{F}U\mathcal{F}^{-1} = M_g$, where $M : L^\infty[0,1] \to B(L^2[0,1])$ is the multiplication operator representation defined in lectures, and $g(t) = e^{-2\pi it}$.

3. (a) Recall from Assignment 1, Question 5, that there is a one-to-one correspondence between (orthogonal) projections in a Hilbert space $\mathcal{H}$ and closed subspaces of $\mathcal{H}$ given by $P \mapsto \text{Range}(P)$. Let P and Q be projections in $B(\mathcal{H})$. Prove that Range(P) and Range(Q) are orthogonal if and only if $PQ = QP = 0$, and in this case $P + Q$ is a projection, with $\text{Range}(P + Q) = \text{Range}(P) \oplus \text{Range}(Q)$.

   (b) Let P and Q be as in part (a). Prove that $\text{Range}(P) \subseteq \text{Range}(Q)$ if and only if $PQ = QP = P$, and in this case $Q - P$ is a projection.