University of Colorado
Department of Mathematics

2016/20017 Semester 2 | Math 8340 Functional Analysis 2 | Assignment 3

Due Friday, March 17, 2017

1. Do Exercises VII.1.1 p. 191, VII.3.4 (over the field $\mathbb{C}$) p.194, and VII.3.1 and VII.3.7 p. 198–199, in the Conway textbook.

2. Recall that $L^1(\mathbb{R})$ and $\ell^1(\mathbb{Z})$ are Banach algebras, where multiplication is defined by convolution. Define an involution on $L^1(\mathbb{R})$ by $f^*(t) = \overline{f(-t)}$ and on $\ell^1(\mathbb{Z})$ by $[(a_n)_{n\in\mathbb{Z}}]^* = (\overline{a_{-n}})_{n\in\mathbb{Z}}$. Prove that $L^1(\mathbb{R})$ and $\ell^1(\mathbb{Z})$ are Banach-$*$ algebras but that they are not C^*-algebras.

3. Let $\mathcal{D}$ be the disk algebra discussed in lectures (recall this can be identified with the norm closure of complex polynomials with non-negative powers of z in $C(\mathbb{T})$). For $f \in \mathcal{D}$, define $f^*(z) = \overline{f(\overline{z})}$. Prove that this defines an involution on $\mathcal{D}$ that makes $\mathcal{D}$ into a Banach $*$-algebra which is not a C^*-algebra.

4. Let $\mathcal{A}$ be a unital C^*-algebra, and suppose that a is a normal element of $\mathcal{A}$.
   (a) Prove that
   $$\|a^2\| = \|a\|^2.$$
   (b) Use mathematical induction to prove that
   $$\|a^{2^n}\| = \|a\|^{2^n}, \forall n \in \mathbb{N}.$$
   (c) Using the spectral radius formula, deduce that
   $$r(a) = \|a\|.$$

5. Let $\mathcal{A}$ be a unital C^*-algebra, and let $\mathcal{B}$ denote a unital C^*-subalgebra of A, so that $\mathcal{B}$ is a $*$-subalgebra of $\mathcal{A}$ which contains the unit of $\mathcal{A}$ and is closed in norm.
   (a) Suppose that b is a self-adjoint element of $\mathcal{B}$ which is invertible in $\mathcal{A}$. Prove that b is invertible in $\mathcal{B}$. [Hint: apply the Gelfand-Naimark Theorem to the C^*-algebra of $\mathcal{A}$ generated by the self-adjoint elements $1_{\mathcal{A}}$, b, and b^{-1}, and use the Weierstrass polynomial approximation theorem to show that b^{-1} can be approximated by a polynomial in b.]

(b) If $w \in \mathcal{B}$ and w is invertible in A, prove that w is invertible in $\mathcal{B}$. [Hint: apply part (a) to the element w^*w.]

(c) Let $a \in \mathcal{B}$. Prove that

$$sp_{\mathcal{B}}(a) = sp_{\mathcal{A}}(a).$$