University of Colorado

Department of Mathematics

2016/20017 Semester 2 Math 8340 Functional Analysis 2 Assignment 4

**Due Monday, April 24, 2017**

1. Do Exercise IX.2.19 p. 267 in the Conway textbook, parts (a)–(h) (not including part (i) ).

2. Let $(X, \Omega)$ be a measurable space, let $\mathcal{H}$ be a complex Hilbert space, and let $E : \Omega \to B(\mathcal{H})$ spectral measure on measure on $(X, \Omega)$. Fix $f \in L^\infty(E)$, the complex-valued measurable functions on $X$ that are essentially bounded with respect to $E$. Let $\{z_n\}_{n=1}^\infty$ be a countable dense subset of $\mathbb{C}$, and define $D \subset \mathbb{N} \times \mathbb{Q}^+$ by

$$D \;=\; \{(n, r) : E(f^{-1}(B(z_n, r))) = 0\},$$

where $B(z_n, r)$ represents the open disk in $\mathbb{C}$ centered at $z_n$ of radius $r > 0$.

   (a) Prove that

$$\{\lambda \in \mathbb{C} : E(f^{-1}(B(\lambda, \epsilon))) \neq\; 0,\; \forall \epsilon > 0\} =\; \mathbb{C} \setminus \cup_{(n,r)\in D} B(z_n, r).$$

   The subset of $\mathbb{C}$ defined in either side of the above equation is called the **essential range** of $f$ with respect to $E$.

   (b) Let $(X, \Omega)$, $f$, $E$ and $\mathcal{H}$ be as above. Prove that the operator $\int_X f(x)\, dE(x)$ is invertible if and only if 0 is not an element of the essential range of $f$ with respect to $E$.

3. Prove using the spectral theorem that if $T$ is a positive compact operator on the complex Hilbert space $\mathcal{H}$, then $\mathcal{H}$ has a orthonormal basis consisting of eigenvectors for $T$.

4. Let $T \in B(\mathcal{H})$ for a complex Hilbert space $\mathcal{H}$. Prove that $T$ is compact if and only if $|T|$ is compact.