University of Colorado
Department of Mathematics

2007/2008 Semester 1 | Math 4310/Analysis | First Midterm Review Sheet

1. Prove that if r_1 and r_2 are rational numbers, then their sum r_1+r_2 and their product $r_1 \cdot r_2$ are rational numbers.

2. Prove that $(5-\sqrt{3})^{\frac{1}{3}}$ does not represent a rational number.

3. Let $a, b, c \in \mathbb{R}$. Show that $|a-b| \leq c$ if and only if $b-c \leq a \leq b+c$.

4. (a) State and prove the Archimedean Property.

   (b) Prove that if $a>0$, then there exists $n \in \mathbb{N}$ such that
   $$\frac{1}{n} < a < n.$$

5. Let S be a bounded non-empty subset of $\mathbb{R}$.
   (a) Define $\sup S$ and $\inf S$.
   (b) Prove that $\inf S \leq \sup S$.
   (c) Let M be an upper bound for S. Prove that $M = \sup S$ if and only if for every $\epsilon > 0$, there exists $s \in S$ (depending on ϵ) such that $M < s + \epsilon$.

6. Determine by inspection $\sup S$ and $\inf S$ for $S = \{-1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$.

7. For each of the following statements, either prove that is is true, or give a counterexample to show that it is false.
   (a) Every bounded subset of rational numbers has a supremum that is rational.
   (b) If (r_n) is a sequence of irrational numbers converging to $r \in \mathbb{R}$, then r is an irrational number.
   (c) If (a_n) and (b_n) are convergent sequences, and if (c_n) is a sequence satisfying
   $$a_n \leq c_n \leq b_n, \ \forall n \in \mathbb{N},$$
   then (c_n) converges.
   (d) If (r_n) and (s_n) are sequences such that (r_n) converges and (s_n) does not converge, then $(r_n + s_n)$ does not converge.

8. Let (s_n) be a sequence in $\mathbb{R}$.
   (a) Give the definition of convergence of (s_n) to $s \in \mathbb{R}$.
   (b) Use the definition of limit of a sequence to give a formal proof that
   $$\lim \frac{4n+3}{5n+2} = \frac{4}{5}.$$
   (c) Prove using limit theorems or otherwise that
   $$\lim \frac{10^n}{n!} = 0.$$
   (d) Find all positive values of b such that $(\frac{b^n}{2^{n+1}})$ is convergent.

9. Find the limit of the following sequences, if they exist. Justify your reasoning.
   (a) $((2 + \frac{1}{n})^2)$;
   (b) $([\frac{n+1}{n}]^{1/n})$;
   (c) $(\frac{\cos 2n}{n^2})$;
   (d) $(\frac{\sqrt{n}-1}{\sqrt{n}+1})$.

10. (a) State, without proof, the Monotone Convergence Theorem for sequences.
    (b) For $n \in \mathbb{N}$, define s_n by $s_1 = 1$ and $s_{n+1} = \sqrt{3 + 2s_n}$, $n \geq 1$. Prove that (s_n) converges, and calculate $\lim s_n$.

11. Let (s_n) be a convergent sequence with $\lim s_n = L$. Fix $K \in \mathbb{N}$, and define a new sequence (x_n) by $x_n = s_{n+K}$. Use the definition to prove that $\lim x_n = L$.