2007/08 Semester 1 | Math 4310 Intro to Analysis | Assignment 10

Some solutions

p. 176, q. 23.2 (d) Here we have the coefficients of the power series given by

$$a_{2n+1} \;=\; \frac{3^n}{\sqrt{n}}$$

and $a_{2n} = 0$. Using the formula in Theorem 23.1, we calculate

$$\begin{aligned}\beta = \limsup|a_k|^{1/k} &= \limsup|a_{2n+1}|^{1/(2n+1)} \\ = \limsup\left|\frac{3^n}{\sqrt{n}}\right|^{1/(2n+1)} &= \limsup[3^{\frac{1}{2+1/n}} \cdot (\frac{1}{n})^{\frac{1}{4n+2}}] \\ = \limsup[\sqrt{3} \cdot [(\frac{1}{n})^{\frac{1}{n}}]^{\frac{n}{4n+2}}] &= \sqrt{3} \cdot \limsup[(\frac{1}{n})^{\frac{1}{n}}]^{\frac{1}{4+2/n}} \\ = \sqrt{3} \cdot 1 &= \sqrt{3}.\end{aligned}$$

It follows that

$$R = \frac{1}{\beta} = \frac{1}{\sqrt{3}}.$$

So the power series converges for $|x| < \frac{1}{\sqrt{3}}$. For $x = \frac{1}{\sqrt{3}}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{3^n}{\sqrt{n}}[\frac{1}{\sqrt{3}}]^{2n+1} = \frac{1}{\sqrt{3}} \sum_{n=0} \frac{1}{\sqrt{n}}$$

which diverges. For $x = -\frac{1}{\sqrt{3}}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{3^n}{\sqrt{n}}[-\frac{1}{\sqrt{3}}]^{2n+1} = -\frac{1}{\sqrt{3}} \sum_{n=0} \frac{1}{\sqrt{n}}$$

which also diverges. Therefore the interval of convergence for the series is the open interval $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

p. 185: q. 24.4 Consider $f_n(x) = \frac{x^n}{1+x^n}$ on the interval $[0, \infty)$. For $x = 0$, $f_n(0) = 0$, for all $n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} f_n(0) = 0.$$

For $x = 1$, $f_n(1) = \frac{1}{2}$, for all $n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} f_n(1) = \frac{1}{2}.$$

For $x \in (0, 1)$, we have

$$0 \le f_n(x) \le x^n, \ \forall n,$$

and since $\lim x^n = 0$ for fixed values of $x \in (0,1)$, we have

$$\lim_{n\to\infty} f_n(x) = 0, \ x \in (0,1).$$

For $x > 1$, $f_n(x) = \frac{1}{1+(1/x^n)}$, and since $\lim(1/x^n) = 0$ for fixed $x > 1$ we have

$$\lim_{n\to\infty} f_n(x) = 1, \ x \in \ (1,\infty).$$

It follows that the limit function is defined by $f : [0,\infty) \to \mathbb{R}$ with $f(x) = 0$ for $0 \le x < 1$, $f(1) = \frac{1}{2}$, and $f(x) = 1$ for $x > 1$. Note that each f_n is continuous on $[0,\infty)$. If (f_n) were to converge to f uniformly on $[0,1]$, it would follow from Theorem 24.3 that f would be continuous on $[0,1]$. But obviously f is not continuous at the point $x = 1$. A similar reasoning shows that (f_n) does not converge to f uniformly on $[0,\infty)$.

p. 185: q. 24.6 Defining $f_n(x) = (x - \frac{1}{n})^2$, it is evident that f_n converges to $f(x) = x^2$ pointwise on $[0,1]$. We now show (f_n) converges to f uniformly on $[0,1]$. Note that

$$|f(x) - f_n(x)| = |x^2 - (x^2 - \frac{2x}{n} + \frac{1}{n^2})| = |\frac{2x}{n} - \frac{1}{n^2}|$$

and that

$$0 \le |\frac{2x}{n} - \frac{1}{n^2}| \le \frac{2x}{n} + \frac{1}{n^2} \le \frac{2}{n} + \frac{1}{n^2} \le \frac{3}{n}$$

for $x \in [0,1]$ and $\lim \frac{3}{n} = 0$ at a rate independent of $x \in [0,1]$. It follows that if $\epsilon > 0$ is given and $n > \frac{3}{\epsilon}$,

$$|f(x) - f_n(x)| \le \frac{3}{n} < \epsilon$$

for all $n > \frac{3}{\epsilon}$ and all $x \in [0,1]$. Thus (f_n) converges to f uniformly on $[0,1]$.

p. 191, q. 25.6 For part (a), use Theorem 25.7, the Weierstrass M-test. Note that for $x \in [-1,1]$,

$$|a_k x^k| = |a_k||x|^k \le \ |a_k| \cdot 1 = \ |a_k|,$$

and $\sum |a_k| < \infty$. Let $M_k = |a_k|$; it then follows by the Weierstrass M-test that $\sum a_k x^k$ converges uniformly on $[-1,1]$. Since the partial sums $S_n(x) = \sum_{k=1}^n a_k x^k$ are polynomials, they are continuous on $[-1,1]$. We thus have a sequence of continuous functons (S_n) that converge uniformly on $[-1,1]$. By Theorem 25.5, the limit function $\sum_{k=1}^\infty a_k x^k$ will be a continuous function.

For part (b), we apply part (a), with $a_k = \frac{1}{k^2}$. Since $\sum_{k=1}^\infty \frac{1}{k^2} < \infty$, by part (a) $\sum_{k=1}^\infty \frac{1}{k^2} x^k$ converges uniformly on $[-1,1]$ to a continuous function. Changing index from k to n, we obtain that $\sum_{n=1}^\infty \frac{1}{n^2} x^n$ represents a continuous function on $[-1,1]$.

p. 211, q. 28.4(b) Here $f(x) = x^2 \sin\frac{1}{x}$ for $x \ne 0$ and $f(0) = 0$. Using the definition of derivative, in order for $f'(0)$ to exist, we need

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

to exist as a real number. We compute:

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0}\frac{x^2\sin\frac{1}{x}-0}{x}\\ &= \lim_{x\to 0} x\sin\frac{1}{x}.\end{aligned}$$

Now $-1 \leq \sin\frac{1}{x} \leq 1$ for all $x \neq 0$, so that

$$-|x| \leq |x\sin\frac{1}{x}| \leq |x|,\ x \neq 0.$$

Since

$$\lim_{x\to 0} -|x| = \lim_{x\to 0}|x| = 0,$$

by the Squeeze Theorem we have

$$\lim_{x\to 0}|x\sin\frac{1}{x}| = 0,$$

so that

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

Thus

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = 0,$$

so that we have proved f is differentiable at $x = 0$ with $f'(0) = 0$.