Some solutions

p. 212, q. 28.6(b) Here $f(x) = x \sin \frac{1}{x}$, $x \neq 0$ and $f(0) = 0$. In order that f be differentiable at $x = 0$, it is necessary and sufficient that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

exist in $\mathbb{R}$. But for $x \neq 0$,

$$\frac{f(x) - f(0)}{x - 0} = \frac{x \sin \frac{1}{x}}{x} = \sin \frac{1}{x},$$

and $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist, as can be seen by examining the value of $g(x) = \sin \frac{1}{x}$ at the two different sequences $(x_n = \frac{1}{2\pi n})$, and $(y_n = \frac{1}{2\pi n + \pi/2})$. Both (x_n) and (y_n) converge to 0 as $n \to \infty$, but $g(x_n) = 0$ for all $n \in \mathbb{N}$ and $g(y_n) = 1$ for all $n \in \mathbb{N}$. Therefore $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist so that f is not differentiable at 0.

p. 212: q. 28.8 We consider part (b). Let $x \neq 0$. We can find a sequence (x_n) of rational numbers such that $\lim x_n = x$, and for each n, $f(x_n) = (x_n)^2$ so that $\lim f(x_n) = \lim x_n^2 = x^2 \neq 0$. On the other hand, for this same value of $x \neq 0$ we can find a sequence (y_n) of irrational numbers with $\lim y_n = x$, and in this case for each $n \in \mathbb{N}$, $f(y_n) = 0$. Thus

$$\lim f(y_n) = 0 \neq x^2 = \lim f(x_n)$$

even though $\lim x_n = x = \lim y_n$. It follows that f is not continuous at x, since we have shown that $\lim_{t \to x} f(t)$ does not exist, and the existence of this limit is a necessary condition for continuity.
For part (c), again we go back to the definition of the derivative at $x = 0$. In order that f be differentiable at $x = 0$, it is necessary and sufficient that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

exist in $\mathbb{R}$. Since 0 is rational, $f(0) = 0^2 = 0$. When x is irrational,

$$\frac{f(x) - f(0)}{x - 0} = \frac{0}{x} = 0.$$

When x is rational,

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^2}{x} = x.$$

In either case, we see that for $x \neq 0$,

$$-|x| \leq |\frac{f(x) - f(0)}{x - 0}| \leq |x|.$$

Since

$$\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0,$$

by the Squeeze Theorem we have

$$\lim_{x\to 0} |\frac{f(x)-f(0)}{x-0}| = 0,$$

so that

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0.$$

Thus

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0,$$

so that we have proved f is differentiable at $x = 0$ with $f'(0) = 0$.

p. 221: q. 29.12 For part (b), let $f(x) = \frac{x}{\sin x}$, $x \in (0, \frac{\pi}{2})$. We consider the sign of the derivative of f on $(0, \frac{\pi}{2})$. By the quotient rule,

$$f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}.$$

Since $\sin^2 x > 0$ for $x \in (0, \frac{\pi}{2})$, in order to show $f'(x) > 0$ for $x \in (0, \frac{\pi}{2})$, it will be sufficient to show that $\sin x - x\cos x > 0$ for $x \in (0, \frac{\pi}{2})$. On $(0, \frac{\pi}{2})$, $\sin x - x\cos x > 0$ if and only if $\frac{\sin x}{\cos x} - x > 0$ if and only if $\tan x - x > 0$. Let $g(x) = \tan x - x$. Then on $(0, \frac{\pi}{2})$, $g'(x) = \sec^2 x - 1$. Let $x \in (0, \frac{\pi}{2})$. Then g is continuous on $[0, x]$ and differentiable on $(0, \frac{\pi}{2}$. By the Mean Value Theorem applied to g, there exists y with $0 < y < x < \frac{\pi}{2}$ and

$$g(x) - g(0) = g'(y) \cdot (x - 0).$$

Note $g(0) = \tan 0 - 0 = 0 - 0 = 0$. Thus

$$\tan x - x - 0 = (\sec^2 y - 1)x$$

for some y with $0 < y < x < \frac{\pi}{2}$. But in this case, $\sec^2 y - 1 > 0$ so that $(\sec^2 y - 1) \cdot x > 0$. It follows that whenever $x \in (0, \frac{\pi}{2})$, $\tan x - x > 0$, and this implies that whenever $x \in (0, \frac{\pi}{2})$, $f'(x) > 0$. By Corollary 29.7, f is strictly increasing on $(0, \frac{\pi}{2})$.

For part (c), if $x = 0$,

$$x = 0 = \frac{\pi}{2} \cdot 0 = \frac{\pi}{2} \cdot \sin 0.$$

So the statement is true for $x = 0$. if $x = \frac{\pi}{2}$,

$$x = \frac{\pi}{2} = \frac{\pi}{2} \cdot 1 = \frac{\pi}{2} \cdot \sin \frac{\pi}{2}.$$

So the statement is true for $x = \frac{\pi}{2}$. For the case where $x \in (0, \frac{\pi}{2})$, consider the function f defined on $[x, \frac{\pi}{2}]$ by $f(x) = \frac{x}{\sin x}$. The function f is continuous on $[0, \frac{\pi}{2}]$ and differentiabe on $(0, \frac{\pi}{2})$. By the Mean Value Theorem, there exists $y \in (0, \frac{\pi}{2})$ such that

$$f(\frac{\pi}{2}) - f(x) = f'(y) \cdot (\frac{\pi}{2} - x).$$

But from the argument of part (b), $f'(y) > 0$ on $(0, \frac{\pi}{2})$, so that $f(\frac{\pi}{2}) - f(x) > 0$ for $x \in (0, \frac{\pi}{2})$. Thus

$$\frac{\pi}{2} \cdot 1 - \frac{x}{\sin x} > 0$$

for $x \in (0, \frac{\pi}{2})$. From this, we derive

$$x < \frac{\pi}{2} \sin x$$

for $x \in (0, \frac{\pi}{2})$. All of this put together gives

$$x \leq \frac{\pi}{2} \sin x, \ x \in [0, \frac{\pi}{2}],$$

as desired.

p. 199, q. 25.3(a) By 29.2(a), $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$. Using Theorem 26.5, we obtain for $|x| < 1$,

$$\frac{d}{dx}[\frac{x}{(1-x)^2}] = \sum_{n=1}^{\infty} n^2 x^{n-1}.$$

Thus by the quotient rule for differentiation,

$$\frac{(1-x)^2 + x \cdot 2(1-x)}{(1-x)^4} = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

for $|x| < 1$. Simplifying, we obtain

$$\frac{1-x^2}{(1-x)^4} = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

for $|x| < 1$, or,

$$\frac{1+x}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

for $|x| < 1$. Multiplying both sides by x, we get

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x+x^2}{(1-x)^3}$$

for $|x| < 1$.