Some solutions

p. 5, q. 1.2 We want to prove that

$$3 + 11 + \cdots (8n - 5) = 4n^2 - n, \ \forall\, n \in \mathbb{N}.$$

Call the above Statement P_n.
We check that P_1 is true. LHS of P_1 is $8 \cdot 1 - 5 = 3$, and RHS of P_1 is $4 \cdot (1)^2 - 1 = 3$.
Therefore $P(_1$ is true, and we have established the basis for our induction.
We now assume P_n is true, and use it to prove that P_{n+1} is true.

$$3 + 11 + \cdots + (8n - 5) + [8(n+1) - 5] = [3 + 11 + \cdots + (8n - 5)] + 8n + 3$$

which, by using the fact that P_n is true,

$$= 4n^2 - n + 8n + 3 = 4(n^2 + 2n + 1) - 4 - n + 3 = 4(n+1)^2 - n - 1$$
$$= 4(n+1)^2 - (n+1).$$

Therefore we have shown that

$$3 + 11 + \cdots + (8n - 5) + [8(n+1) - 5] = 4(n+1)^2 - (n+1),$$

so that P_{n+1} is true.
We have established the induction step, so that P_n is true for all $n \in \mathbb{N}$.

p. 12: q. 2.4 Suppose that b is rational. The rationals are closed under multiplication, so that $b^2 = b \cdot b$ is rational. Similarly, (-7) is rational so that $(-7) \cdot b^2 = -7b^2$ is rational. Finally, 4 is rational, and the rationals are closed under addition, so that $4 + (-7b^2) = 4 - 7b^2$ is rational.

p. 18: q. 3.5 (a) Suppose that $-a \le b \le a$. Then $-a \le b$ and $b \le a$. If b is non-negative, we have $|b| = b$, and in this case $|b| = b \le a$. If b is negative, then $|b| = -b$. In this case, if we multiply the inequality $-a \le b$ by (-1) we get $(-1) \cdot (-a) \ge (-1)b$, that is, $a \ge -b$. So if b is negative we also have $|b| = -b \le a$. In both cases we obtain $|b| \le a$.
Now suppose that $|b| \le a$. Since $|b|$ is non-negative, we know that $0 \le |b| \le a$ so that $0 \le a$, and multiplying both sides of the latter inequality by (-1) we get $-a \le 0$.
If $b \ge 0$, then $b = |b| \le a$ so that $b \le a$. Also, since $0 \le -a$, using the fact that $b \ge 0$ gives $-a \le b$. Hence for $b \ge 0$ we get $-a \le b \le a$.
If $b \le 0$, then $|b| = -b$, so the statement that $|b| \le a$ means $-b \le a$. Multiplying both sides by (-1) gives $b \ge -a$. The fact that $b \le 0$ and $0 \le a$ means that in addition $b \le a$. Hence $-a \le b \le a$ for the case $b \le 0$ as well.

(b) Note that $|a| = |a + 0| = |a - b + b| = |(a - b) + b|$. By the triangle inequality, $|(a - b) + b| \leq |a - b| + |b|$. Hence $|a| \leq |a - b| + |b|$ giving

$$|a| - |b| \leq |a - b|. \ (1)$$

If we interchange the roles of b and a in the above argument we get

$$|b| - |a| \leq |b - a|. \ (2)$$

But $|b - a| = |-(a - b)| = |a - b|$. It follows that $|b| - |a| \leq |a - b|$. We multiply both sides of the last inequality by (-1) to get

$$-|b| + |a| \geq -|a - b|. \ (3)$$

It follows from (1) and (3) that

$$-|a - b| \leq |a| - |b| \leq |a - b|.$$

We now use part (a) of 3.5 to deduce that

$$||a| - |b|| \ \leq \ |a - b|.$$

p. 19: q. 3.7 We are given that a, b are fixed real numbers, and we are told that whenever b_1 is any real number with $b_1 > b$, then $a \leq b_1$. We want to prove that $a \leq b$.
Suppose, by way of contradiction, that it is not true that $a \leq b$. Then it must be the case that $b < a$. Using our order axioms, we deduce first that $a + b < a + a = 2a$, and then, since $[1/2] > 0$, that $[1/2] \cdot (a + b) < [1/2] \cdot (2a)$, that is to say, $\frac{a+b}{2} < a$. In a similar fashion, we can show that $b < \frac{a+b}{2}$.
Now let $b_1 = \frac{a+b}{2}$. We have shown that $b_1 > b$. Yet it is not true that $a \leq b_1$, in fact, we have shown that $b_1 < a$. This contradicts our given information. Therefore, what we assummed, that $b < a$, must be false, since it lead us to a contradiction. It follows that $a \leq b$, as desired.