2007/08 Semester 1 Math 4310 Intro to Analysis Assignment 2

Some solutions

p. 25, q. 4.6 (a) Since S is not empty, we can find an element $s \in S$. Since S is bounded, it is bounded below, and $inf(S)$ is a lower bound for S, so that

$$inf(S) \leq s.$$

Also, S is bounded above, and $sup(S)$ is an upper bound for S, so that

$$s \leq sup(S).$$

Using the transitivity of inequalities, we obtain

$$inf(S) \leq sup(S).$$

(b) Since S be non empty. Again we can say there exists $s \in S$. Take an arbitrary $s \in S$. The argument above showed that

$$inf(S) \leq s \leq sup(S).$$

But we are given that $inf(S) = sup(S)$. So

$$sup(S) \leq s \leq sup(S),$$

so that we must have $s = sup(S)$.
Since we chose s to be an arbitray element of S, it follows that S consists of the one-point set, $S = \{sup(S) = inf(S)\}$.

p. 26: q. 4.7 (b) We first show that

$$sup(S \cup T) \leq \max\{sup(S), sup(T)\}.$$

Let r be an element of $S \cup T$. If $r \in S$, then $r \leq sup(S)$ since $sup(S)$ is an upper bound for S. Also $sup(S) \leq \max\{sup(S), sup(T)\}$, by definition of max. It follows that

$$r \leq \max\{sup(S), sup(T)\}.$$

If $r \in T$, one shows in a similar fashion that

$$r \leq \max\{sup(S), sup(T)\}.$$

It follows that $\max\{sup(S), sup(T)\}$ is an upper bound for $S \cup T$, so that

$$sup(S \cup T) \leq \max\{sup(S), sup(T)\}, \quad (1)$$

by definition of sup. We now show that

$$\max\{sup(S), sup(T)\} \leq sup(S \cup T).$$

Using the fact that $sup(S \cup T)$ is the least upper bound for $S \cup T$, we have that

$$sup(S \cup T) \geq r,\ \forall r \in S \cup T.$$

This means

$$sup(S \cup T) \geq s,\ \forall s \in S$$

and

$$sup(S \cup T) \geq t,\ \forall t \in T.$$

It follows that $sup(S \cup T)$ is an upper bound for both S and T.
But then,

$$sup(S \cup T) \geq sup(S)$$

and

$$sup(S \cup T) \geq sup(T),$$

by definition of least upper bound. It follows that

$$sup(S \cup T) \geq \ \max\{sup(S), sup(T)\}.\ \ (2).$$

Inequalities (1) and (2) give

$$sup(S \cup T) = \max\{sup(S), sup(T)\}.$$