2007/08 Semester 1 — Math 4310 Intro to Analysis — Assignment 3

Some solutions

p. 26, q. 4.10 We use Corollaries of the Archimedean property. By Property (**) on p. 23, there is $n_1 \in \mathbb{N}$ such that $a < n_1$. By Property (*) on p. 23, there is $n_2 \in \mathbb{N}$ such that $\frac{1}{n_2} < a$. Now let $n = \max\{n_1, n_2\}$, Then $n \geq n_1$ and $n \geq n_2$ so that $\frac{1}{n} \leq \frac{1}{n_2}$. It follows that

$$\frac{1}{n} < \ a < \ n,$$

as desired.

p. 26: q. 4.12 We first show that given $r \in \mathbb{Q}$, $N \in \mathbb{N}$, $r + \frac{\sqrt{2}}{N}$ is irrational. Recall $\sqrt{2}$ is irrational. It follows that $s = \frac{\sqrt{2}}{N}$ is irrational for every $N \in \mathbb{N}$. (If s_N were rational, $N \cdot s)N = \sqrt{2}$ would be rational, a contradiction). Thus $t_N = r + \frac{\sqrt{2}}{N}$ is irrational for each $N \in \mathbb{N}$ (if t_N were rational, $t_N - r = s_N$ would also be rational, which we have shown is not the case). Now suppose we are given two real numbers a and b with $a < b$. By Denseness of $\mathbb{Q}$ 4.7, we can find a rational number r with $a < r < b$, so that $b - r > 0$. Let $a' = (b - r) > 0$, and let $b' = \sqrt{2} > 0$. By the Archimedean property, there is some $N \in \mathbb{N}$ with $Na' > b'$. Thus there exists $N \in \mathbb{N}$ with $N(b-r) > \sqrt{2}$. Thus there is some $N \in \mathbb{N}$ with $b - r > \frac{\sqrt{2}}{N}$. Thus there exists some $N \in \mathbb{N}$ with $b > r + \frac{\sqrt{2}}{N}$. We have therefore proved the existence of $N \in \mathbb{N}$ with

$$a < r < r + \frac{\sqrt{2}}{N} < b.$$

Recalling that $r + \frac{\sqrt{2}}{N}$ is irrational, we have found irrational $x = r + \frac{\sqrt{2}}{N}$ with $a < x < b$.

p. 36: q. 7.4 (a) Let $(x_n = \frac{\sqrt{2}}{n})$. The argument given in q. 4.12 shows that x_n is irrational for every $n \in \mathbb{N}$. But

$$\lim x_n \ = \ 0,$$

and 0 is a rational number.

(b) Let an irrational number x be given. By Property 4.7 (Density of Rationals), for each $n \in \mathbb{N}$ we can find a rational number r_n with $x < r_n < x + \frac{1}{n}$. By the Squeeze principle,

$$\lim r_n \ = \ x.$$

p. 42: q. 8.1 (c) let $\epsilon > 0$ be given. We want to find $N > 0$ such that whenever $n > N$,

$$|\frac{2n-1}{3n+2} - \frac{2}{3}| < \epsilon.$$

We use algebra to write:

$$\frac{2n-1}{3n+2} - \frac{2}{3} = \frac{3(2n-1)}{3(3n+2)} - \frac{2(3n+2)}{3(3n+2)}$$

$$= \frac{-7}{3(3n+2)}.$$

So for all $n \in \mathbb{N}$,

$$|\frac{2n-1}{3n+2} - \frac{2}{3}| = |\frac{-7}{3(3n+2)}| = \frac{7}{9n+6} < \frac{7}{9n}.$$

Note if $n > N > 0$, we have $\frac{7}{9n} < \frac{7}{9N}$. So we just need to choose $N > 0$ so that

$$\frac{7}{9N} \leq \epsilon,$$

and then if $n > N$, we will have by stringing the inequalities together

$$|\frac{2n-1}{3n+2} - \frac{2}{3}| < \frac{7}{9n} < \frac{7}{9N} \leq \epsilon.$$

By algebra we solve that $\frac{7}{9N} \leq \epsilon$ if $N \geq \frac{7}{9\epsilon}$. So choose $N = \frac{7}{9\epsilon}$. Then whenever $n > N$,

$$|\frac{2n-1}{3n+2} - \frac{2}{3}| < \epsilon.$$

This completes the formal proof that

$$\lim \frac{2n-1}{3n+2} = \frac{2}{3}.$$