Some solutions

p. 42, q. 8.2 (b) We want to give a formal proof that

$$\lim \frac{7n-19}{3n+7} \;=\; \frac{7}{3}.$$

We must show that if $\epsilon > 0$ is given, there exists $N \in \mathbb{N}$ such that whenever $n > N$,

$$|\frac{7n-19}{3n+7} - \frac{7}{3}| < \epsilon.$$

We note that for all $n \in \mathbb{N}$,

$$|\frac{7n-19}{3n+7} - \frac{7}{3}| = |\frac{3(7n-19)}{3(3n+7)} - \frac{7(3n+7)}{3(3n+7)}|$$

$$= |\frac{21n-57-21n-49}{9n+21}| = |\frac{-106}{9n+21}| < \frac{106}{9n}.$$

Now if $n > N > 0$, we know that $\frac{1}{n} < \frac{1}{N}$ so that $\frac{106}{9n} < \frac{106}{9N}$. Hence, if we choose N so that $\frac{106}{9N} \le \epsilon$, whenever $n > N$, it will follow that $\frac{106}{9n} < \epsilon$. Thus, choose $N \ge \frac{106}{9\epsilon}$. For $n > N$ we have:

$$|\frac{7n-19}{3n+7} - \frac{7}{3}| < \frac{106}{9n} < \frac{106}{9N} < \epsilon.$$

It follows that $n > N$ implies that

$$|\frac{7n-19}{3n+7} - \frac{7}{3}| < \epsilon,$$

so that we have given a formal proof that

$$\lim \frac{7n-19}{3n+7} \;=\; \frac{7}{3}.$$

p. 42: q. 8.5 (a) We are given sequences (a_n), (b_n), and (s_n) with

$$a_n \;\le\; s_n \;\le\; b_n, \; \forall n \in \mathbb{N},$$

and

$$\lim a_n \;=\; \lim b_n \;=\; s.$$

We want to prove that

$$\lim s_n = s.$$

Thus given $\epsilon > 0$ we need to find $N \in \mathbb{N}$ such that whenever $n > N$,

$$|s_n - s| \;<\; \epsilon.$$

First, since $\lim a_n = s$, we can find $N_1 \in \mathbb{N}$ such that whenever $n > N_1$,

$$|a_n - s| < \epsilon.$$

Thus whenever $n > N_1$,

$$-\epsilon < a_n - s < \epsilon,$$

or

$$s - \epsilon < a_n < s + \epsilon \ \ (1).$$

Similarly, since $\lim_b n = s$, we can find $N_2 \in \mathbb{N}$ such that whenever $n > N_2$,

$$|a_n - s| < \epsilon.$$

Thus whenever $n > N_2$,

$$-\epsilon < b_n - s < \epsilon,$$

or

$$s - \epsilon < b_n < s + \epsilon. \ \ (2)$$

Now let $N = \max\{N_1, N_2\}$. If $n > N$, we must have $n > N_1$ and $n > N_2$. Thus for every $n > N$ we have by (1)

$$s - \epsilon < a_n$$

and by (2)

$$b_n < s + \epsilon.$$

Since for every n we know that $a_n \leq s_n \leq b_n$, it follows that for all $n > N$,

$$s - \epsilon < a_n \leq s_n \leq b_n < s + \epsilon.$$

Thus whenever $n > N$,

$$s - \epsilon < s_n < s + \epsilon,$$

or

$$-\epsilon < s_n - s < \epsilon.$$

Thus is the same as saying that whenever $n > N$,

$$|s_n - s| < \epsilon,$$

and since ϵ was arbitrary, we have shown that

$$\lim s_n = s.$$

p. 52: q. 9.4(a) This was done in class on 9/24, but we review the answer. Recall one can prove by induction first that the sequence (s_n) defined recursively by $s_1 = 1$ and $s_{n+1} = \sqrt{s_n + 1}$, $n \geq 1$ is non-negative, non-decreasing, and is bounded from above by 2, for example. Thus by the Monotone Convergence Theorem for sequences, the sequence (s_n) must converge to $s \in \mathbb{R}$. Also note that s will be non-negative since each of the s_n in non-negative.

Since for all $n \in \mathbb{N}$ we have $s_{n+1} = \sqrt{s_n + 1}$, we can take the limit of both sides to get

$$\lim s_{n+1} = \lim \sqrt{s_n + 1}.$$

But

$$\lim s_{n+1} = \lim s_n = s,$$

and by limit rules,

$$\lim \sqrt{s_n + 1} = \sqrt{\lim(s_n + 1)} = \sqrt{s + 1}.$$

It follows that

$$s = \sqrt{s + 1},$$

so that

$$s^2 = s + 1$$

and $s^2 - s - 1 = 0$. Solving the quadratic equation we get

$$s = \frac{1 \pm \sqrt{5}}{2}.$$

Since we know s is non-negative, the only possible solution is

$$s = \frac{1 + \sqrt{5}}{2}.$$

p. 53: q. 9.10(a) We are given that

$$\lim s_n = +\infty,$$

and we want to show that if $k > 0$,

$$\lim ks_n = +\infty.$$

Fix $M > 0$. Since $\lim s_n = +\infty$, by definition, we can find $N \in \mathbb{N}$ such that whenever $n > N$,

$$s_n > \frac{M}{k}.$$

But then for $n > N$, if we multiply both sides of the above inequality by the postive number k, we get

$$ks_n > k \cdot \frac{M}{k} = M.$$

It follows that for $n > N$,

$$ks_n > M,$$

and thus by definition,

$$\lim ks_n = +\infty.$$