University of Colorado
Department of Mathematics

2007/2008 Semester 1 Math 4310 Introduction to Analysis Partial Review for Final

Please go over review sheets for both midterms, too.

Some solutions

1. For each of the following statements, either prove that it is true, or give a counterexample to show that it is false.

   (a) Let $x_0 \in (a,b)$. If $f(x) < g(x)$ for all $x \in (a,b)$ and $\lim_{x\to x_0} f(x) = L_1$ and $\lim_{x\to x_0} g(x) = L_2$ both exist and are finite, then $L_1 < L_2$.
   FALSE – Let $(a,b) = (-1,1)$, $f(x) \equiv 0$, $g(x) = |x|$ for $x \neq 0$ and $g(0) = 1$. Then
   $$\lim_{x\to 0} f(x) = 0 \ = \ \lim_{x\to 0} g(x).$$

   (b) If the sequence of functions (f_n) converges pointwise to a limit function f on a subset S of $\mathbb{R}$, then it converges uniformly on S to the limit function f.
   FALSE – See counterexample in solution to 1 (f) below.

   (c) If the sequence of functions (f_n) converges uniformly to a limit function f on a subset S of $\mathbb{R}$, then it converges pointwise on S to the limit function f.
   TRUE – check relevant definitions.

   (d) If a sequence of bounded functions (f_n) converges uniformly on a subset S of $\mathbb{R}$ to the limit function f, then f is bounded on S.
   TRUE – take $\epsilon = 1$ in the definition of uniform convergence, and find $N \in \mathbb{N}$ such that whenever $n > N$, $|f(x) - f_n(x)| < 1$, $\forall x \in S$. In particular, taking $n = N+1$,
   $$|f(x)| - |f_{N+1}(x)| \ \leq \ |f(x) - f_{N+1}(x)| \ < \ 1, \ \forall x \in S.$$
   It follows that
   $$|f(x)| \ \leq \ 1 + |f_{N+1}(x)|, \ \forall x \in S.$$
   Let $M > 0$ be such that $|f_{N+1}(x)| \leq M$ for all $x \in S$. Then
   $$|f(x)| \ \leq \ 1 + M, \ \ \forall x \in S,$$
   so that f is bounded on S.

   (e) If a sequence of bounded functions (f_n) converges pointwise on a subset S of $\mathbb{R}$ to the limit function f, then f is bounded on S.
   FALSE – Let $S = (0,1)$, let $f_n(x) = n,\ x \in (0, 1/n),\ f_n(x) = 1/x,\ x \in [1/n, 1)$. Then each f_n is bounded by n. Also the (f_n) converge pointwise to $f(x) = 1/x,\ x \in (0,1)$. But f is unbounded.

(f) If a sequence of functions (f_n) are all differentiable on (a, b) and converge pointwise to a limit function f on (a, b), then f is differentiable on (a, b).
FALSE – Let $f_n(x) = x^n$ on $[0, 1]$. Each of the functions f_n is continuous and differentiable on $(0, 1)$ and continuous and differentiable from the right and left at 0 and 1 respectively. However, the limit function is given by $f(x) = 0,\ x \in [0, 1),\ f(1) = 1$, and this function is not continuous (hence not differentiable from the left) at 1.

(g) If a sequence of functions (f_n) are all continuous at x_0 in (a, b) and converge pointwise to a limit function f on (a, b), then f is continuous at x_0.
FALSE – the counterexample in the proceeding question works for this one too.

(h) If a sequence of functions (f_n) are all continuous at $x_0 \in (a, b)$ and converge uniformly to a limit function f on (a, b), then f is continuous at x_0.
TRUE – see proof in Theorem 24.3 on p.180-181.

(i) If a sequence of functions (f_n) are all continuous on $[a, b]$ and converge pointwise to the continuous function f on $[a, b]$, then $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.
FALSE – consider the sequence of hat functions f_n defined for each n to be linear going from 0 to n on $[0, 1/(2n)]$, linear going from n to 0 on $[1/(2n), 1/n]$, constantly equal to 0 on $[1/n, 1]$. Then (f_n) converges pointwise to the function f that is identically 0 on $[0, 1]$. But $\int_0^1 f_n(x)dx = 1/2$ for all n, while the integral of the 0 function from 0 to 1 is 0.

(j) If $f : [a, b] \to \mathbb{R}$, and f is continuous at $x_0 \in (a, b)$, then f is differentiable at x_0.
FALSE – Consider $f(x) = |x|$, which is continuous at 0 but not differentiable at 0.

(k) If $f : [a, b] \to \mathbb{R}$, and f is differentiable at $x_0 \in (a, b)$, then f is continuous at x_0.
TRUE – See Theorem 28.2, p. 207.

2. Calculate

$$\lim_{x\to+\infty} \frac{\sqrt{1+4x^2}-1}{x^2}.$$

Solution:

$$\lim_{x\to+\infty} \frac{\sqrt{1+4x^2}-1}{x^2} = \lim_{x\to+\infty} \frac{\sqrt{1+4x^2}-1}{x^2} \cdot \frac{\sqrt{1+4x^2}+1}{\sqrt{1+4x^2}+1}$$

$$= \lim_{x\to+\infty} \frac{1+4x^2-1}{x^2(\sqrt{1+4x^2}+1)} = \lim_{x\to+\infty} 4x^2x^2(\sqrt{1+4x^2}+1)$$

$$= \lim_{x\to+\infty} \frac{4}{\sqrt{1+4x^2}+1} = 0.$$

3. If $f(x) \leq g(x) \leq h(x)$ for all $x \in (a,b)$ and $\lim_{x\to a+} f(x) = \lim_{x\to a+} h(x) = L$ for some real number L, prove that $\lim_{x\to a+} g(x) = L$.
   Let $\epsilon > 0$ be given. Find $\delta_1 > 0$ such that whenever $0 < x - a < \delta_1$, $|f(x) - L| < \epsilon$ so that
$$L - \epsilon < \ f(x) \ < L + \epsilon.$$
   Then, find $\delta_2 > 0$ such that whenever $0 < x - a < \delta_2$, $|h(x) - L| < \epsilon$ so that
$$L - \epsilon < \ h(x) \ < L + \epsilon.$$
   Let $\delta = \min\{\delta_1, \delta_2\}$. Then whenever $0 < x - a < \delta$, we have $0 < x - a < \delta_1$ and $0 < x - a < \delta_2$, so that
$$L - \epsilon \ < f(x) \ \leq g(x) \ \leq h(x) < L + \epsilon.$$
   It follows that whenever $0 < x - a < \delta$,
$$|g(x) - L| < \epsilon,$$
   so that $\lim_{x\to a+} g(x) = L$.
4. Let (f_n) be a sequence of functions defined on a non-empty set $S \subseteq \mathbb{R}$. Let f be a function defined on S. Give the definition of (f_n) converging to f pointwise on A. Give the definition of (f_n) converging to f uniformly on S. Give an example of a sequence of functions (f_n) defined on $S \subseteq R$ where (f_n) converges to a limit function f pointwise on A but not uniformly on A.
   The relevant definitions are on p. 177, 178 of the textbook. See counterexample in 1(f) above.
5. Suppose that (f_n) and (g_n) are sequences of continuous functions on $[a,b]$ that converge uniformly to the limit functions f and g, respectively. Prove that the sequence $(f_n + g_n)$ converges uniformly on $[a,b]$ to the limit function $f + g$.
   Fix $\epsilon > 0$. Find $N_1 > 0$ such that whenever $n > N_1$,
$$|f_n(x) - f(x)| < \frac{\epsilon}{2}, \ \forall x \in [a,b].$$
   Then, find $N_2 > 0$ such that whenever $n > N_2$,
$$|g_n(x) - g(x)| < \frac{\epsilon}{2}, \ \forall x \in [a,b].$$
   Let $N = \max\{N_1, N_2\}$. Then if $n > N$, $n > N_1$ and $n > N_2$. It follows that for $n > N$,
$$\begin{aligned} |(f_n + g_n(x)) - (f+g)(x)| \ &= \ |f_n(x) - f(x) + g_n(x) - g(x)| \\ \leq \ |f_n(x) - f(x)| + |g_n(x) - g(x)| \ &< \ \frac{\epsilon}{2} + \frac{\epsilon}{2}, \ \forall x \in [a,b]. \end{aligned}$$
   Thus for $n > N$,
$$|(f_n + g_n(x)) - (f+g)(x)| \ < \ \epsilon, \ \forall x \in [a,b].$$
   It follows that $(f_n + g_n)$ converges uniformly to $f + g$ on $[a,b]$.

6. Let (f_n) be a sequence of continuous functions that converges to f uniformly on the interval $[a, b]$, and suppose that $(x_n) \subseteq [a, b]$ converges to $x_0 \in [a, b]$. Prove that $\lim f_n(x_n) = f(x_0)$.

   Fix $\epsilon > 0$. Since (f_n) converges uniformly to f on $[a, b]$, there exists N_1 such that whenever $n > N_1$,
$$|f_n(x) - f(x)| < \frac{\epsilon}{2}, \ \forall\, x \in [a, b].$$
   Also, since the f_n are all continuous and converge to f uniformly on $[a, b]$, by Theorem 24.3, the limit function f is continuous on $[a, b]$. In particular f is continuous at x_0. Therefore, there exists $\delta > 0$ such that whenever $|x - x_0| < \delta$ and $x \in [a, b]$,
$$|f(x) - f(x_0)| < \frac{\epsilon}{2}.$$
   Finally, since $\lim x_n = x_0$, there exists $N_2 > 0$ such that whenever $n > N_2$,
$$|x_n - x_0| < \delta.$$
   Let $N = \max\{N_1, N_2\}$. Then if $n > N$, $n > N_1$ and $n > N_2$. It follows that for $n > N$,
$$\begin{aligned} |f_n(x_n) - f(x_0)| &= |f_n(x_n) - f(x_n) + f(x_n) - f(x_0)| \\ &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$
   Thus whenever $n > N$,
$$|f_n(x_n) - f(x_0)| < \epsilon,$$
   so that $\lim f_n(x_n) = f(x_0)$.

7. If $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$, and $f(0) = 0$, compute $f'(x)$ for all real x, and show that f'' does not exist at 0.

   $f'(x) = 2x \sin 1/x + x^2(\cos 1/x)(-1/x^2) = 2x \sin 1/x - \cos 1/x$ for $x \neq 0$. For $a = 0$, we calculate
$$\lim_{x \to 0}(f(x) - f(0))/(x - 0) = \lim_{x \to 0} x \sin 1/x.$$
   But
$$-|x| \leq x \sin 1/x \leq |x|, \ x \neq 0,$$
   so by the Squeeze Theorem we see
$$\lim_{x \to 0} x \sin 1/x = 0.$$
   Thus f is differentiable at 0 with $f'(0) = 0$.

   To check whether or not f' is differentiable at 0 we consider the quotient $(f'(x) - f'(0))/(x - 0)$ for x not equal to 0 and see if it has a limit as x goes to 0.
$$\lim_{x \to 0}(f'(x) - f'(0))/(x - 0) = \lim_{x \to 0} \frac{2x sin 1/x - cos 1/x}{x}$$

$$= \lim_{x \to 0} [2sin1/x - \frac{cos1/x}{x}].$$

But this last quotient has no limit as x goes to 0. Consider the two sequences $(x_n = 1/(2\pi n))$ and $(y_n = 1/[2\pi n + \pi/2])$. Both (x_n) and (y_n) tend to 0 as n goes to infinity. Note $2sin(1/x_n) - \frac{cos1/x_n}{x_n} = -2\pi n$ goes to $+\infty$, and $2sin1/y_n - \frac{cos1/y_n}{y_n} = 2$ for all n. Hence no limit exists, and f' is not differentiable at 0, i.e. $f''(0)$ does not exist.

8. Suppose $f(x) = \begin{cases} 0 & x \text{ rational}, \\ x^2 & x \text{ irrational}. \end{cases}$

   Show that f is continuous at one point only, differentiable at one point only, and compute the derivative of f at that point.

   If $a \neq 0$, find sequences of rational numbers (x_n) and irrational numbers (y_n) with $\lim x_n = \lim y_n = a$ (we can do this by density of rationals and irrationals in $\mathbb{R}$. Then $\lim f(x_n) = \lim 0 = 0$, and $\lim f(y_n) = \lim (y_n)^2 = a^2 \neq 0$. Thus f is not continuous at a, since the definition on p. 116 of the text is not satisfied. By Theorem 28.2, it follows that f is not differentiable at a for $a \neq 0$. Now at $a = 0$, we have $|\frac{f(x)-f(0}{x-0}| \ \leq \ |\frac{x^2}{x}| = |x|$ for $x \neq 0$. Hence

   $$0 \leq \ |\frac{f(x) - f(0}{x - 0}| \ \leq \ |x|, \ x \neq 0.$$

   We know $\lim_{x \to 0} |x| = 0$, so by the Squeeze Theorem $\lim_{x \to 0} |\frac{f(x)-f(0}{x-0}| = 0$, so that $\lim_{x \to 0} \frac{f(x)-f(0}{x-0} = 0$. Thus f is differentiable at 0 with $f'(0) = 0$. By Theorem 28.2, f is also continuous at 0.

9. Let $p(x) = 5x^6 + 6x^5 + C$, where C is a fixed constant. Prove that there exists at most one value $r \in (-\infty, -1]$ such that $p(r) = 0$. [Hint: consider the sign of the derivative of p on $(-\infty, -1]$.]

   Consider $p'(x) = 30x^5 + 30x^4 = 30x^4(x + 1)$. Suppose by way of contradiction that there exists $r_1 < r_2 \leq -1$ such that $p(r_1) = p(r_2) = 0$. By Rolle's' Theorem, there exists y with $r_1 < y < r_2$ such that $p'(y) = 0$. Note $y < -1$ since r_2 is less than or equal to -1 and $y < -1$. But for all $y < -1$ we can see by observation that $p'(y) = 30y^4(y + 1) < 0$, since the product of a positive and negative number is negative. This gives us the desired contradiction.

10. (a) State, without proof, Rolle's Theorem.

    Let f be a continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = g(b)$. There exists [at least one] $x \in (a, b)$ such that $f'(x) = 0$.

    (b) Let f be a function on $\mathbb{R}$ which at least 2 times differentiable, i.e. the derivatives $f'(x)$ and $f''(x)$ exist for all $x \in R$. Suppose that

    $$f(0) = f(1) = f(2) = 0.$$

    Show that there exists $x_0 \in \mathbb{R}$ such that $f''(x_0) = 0$.

    By Rolle's Theorem we can find $x_1 \in (0, 1)$ such that $f'(x_1) = 0$. By Rolle's

Theorem applied to $[1,2]$ we can find x_2 such that $f'(x_2) = 0$. Note $x_1 < x_2$, and f' is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By Rolle's Theorem, we can find $x_0 \in (x_1, x_2)$ such that $f''(x_0) = (f')'(x_0) = 0$.

11. (a) State, without proof, the Mean Value Theorem.
See the statement of Theorem 29.3 p. 215 of the textbook.

(b) Prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
Let $f(x) = \sin x$ and suppose $x < y$. Then f is continuous on $[x, y]$ and differentiable on (x, y) so there exists $x_0 \in (x, y)$ such that

$$\sin x - \sin y = \cos x_0 (x - y).$$

Hence

$$|\sin x - \sin y = \cos x_0 (x - y)| \leq 1 \cdot |x - y| = |x - y|.$$

A similar argument works if $y < x$. If $x = y$, both sides of the desired inequality are equal to 0.

12. (a) State, without proof, Taylors's Theorem.
See Theorem 31.3 on p. 232 of the text.

(b) Find the Taylor series for $\sinh x = \frac{e^x - e^{-x}}{2}$, and prove that it converges to $\sinh x$ for all $x \in \mathbb{R}$.
Note $f'(x) = (e^x + e^{-x})/2, f''(x) = (e^x - e^{-x})/2$, etc., so that

$$f^{(2k-1)}(x) = \cosh x = (e^x + e^{-x})/2, \ k \in \mathbb{N},$$

and

$$f^{(2k}(x) = \sinh x = (e^x - e^{-x})/2, \quad k \in \mathbb{N}.$$

Note that for specific values of $x \in \mathbb{R}$, $|f^n(t)| \leq e^{|x|}, -|x| < t < |x|$. By Taylor's Theorem, for each $n \in \mathbb{N}$,

$$\sinh x = \sum_{k \text{ odd } k \leq n-1} \frac{x^k}{k!} + \frac{f^{(n)}(y)}{n!} x^n,$$

for some y between 0 and x. But

$$\left|\frac{f^{(n)}(y) x^n}{n!}\right| \leq \frac{e^{|x|}|x|^n}{n!}$$

goes to 0 as n goes to infinity for fixed x.
It follows that

$$\sinh x = \sum_{n=1}^{\infty} x^{2n-1}/(2n-1)!$$

13. (p. 199, 27.4 (b)) Using the fact that $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$, express $F(x) = \int_0^x e^{-t^2} dt$ as a power series. Be sure to justify your reasoning.
Note for any $t \in \mathbb{R}$, e^{-t^2} can be expressed as the power series

$$e^{-t^2} = \sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!}.$$

If $x \in \mathbb{R}$ is fixed, this power series converges uniformly in the interval $[-|x|, |x|]$. By our theorems about uniform convergence and integration, it follows that for all $x \in \mathbb{R}$,

$$\begin{aligned} \int_0^x (e^{-t^2}) dt &= \int_0^x \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!} dt = \sum_{k=0}^{\infty} \int_0^x \frac{(-1)^k t^{2k}}{k!} dt \\ &= \sum_{k=0}^{\infty} [\frac{(-1)^k t^{2k+1}}{k!(2k+1)}]_{t=0}^{t=x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!(2k+1)}. \end{aligned}$$

14. (p. 252, 32.3) Define $g : [0, b] \to \mathbb{R}$ by

$$g(x) = \begin{cases} x^2 & x \ \text{ rational,} \\ 0 & x \text{ irrational} \end{cases}$$

Calculate the upper and lower Darboux integrals for g on the interval $[0, b]$, and determine whether or not g is integrable on $[0, b]$.
Recall $g(x) = x^2$ for x rational and $g(x) = 0$ for x irrational on $[0, b]$. We consider the auxiliary function $f(x) = x^2$ on $[0, b]$. Let $\mathcal{P} = \{t_0 = 0 < t_1 < t_2 < \cdots < t_n = b\}$ be an arbitrary partition of $[0, b]$. Then by density of the irrational numbers in $[0, b]$, $\inf \{g(x) : t_{k-1} \leq x \leq t_k\} = 0$. It follows that the lower sum $L(g, \mathcal{P}) = \sum_{k=1}^{n} 0 \cdot (t_k - t_{k-1}) = \sum_{k=1}^{n} 0 = 0$. It follows that the lower integral $L(g) = \sup \{L(g, \mathcal{P}\} = 0$. On the other hand, by density of rationals in $[0, b]$, $\sup \{g(x) : t_{k-1} \leq x \leq t_k\} = (t_k)^2 = \sup \{f(x) : t_{k-1} \leq x \leq t_k\}$. Thus $U(g, \mathcal{P}) = \sum_{k=1}^{n} (t_k)^2 \cdot (t_k - t_{k-1}) = U(f, \mathcal{P})$. Thus the upper integral of g is given by

$$U(g) = \inf \{U(g, \mathcal{P}\} = \inf \{U(f, \mathcal{P}\} = U(f).$$

But, since f is continuous on $[0, b]$, it is integrable on $[0, b]$, and by the fundamental theorem of calculus

$$U(f) = \int_0^b x^2 dx = [\frac{x^3}{3}]_{x=0}^{x=b} = \frac{b^3}{3}.$$

Thus for $b > 0$,

$$L(g) = 0 < \frac{b^3}{3} = U(f) = U(g).$$

It follows that g is not integrable on $[0, b]$.

15. (p. 253, 32.6) Let f be a bounded function on $[a,b]$. Suppose there exist sequences (U_n) and (L_n) of upper and lower Darboux sums for f such that $\lim(U_n - L_n) = 0$. Show that f is integrable on $[a,b]$ and that $\int_a^b f(x)dx = \lim U_n = \lim L_n$.
We are given families (L_n) and (U_n) of lower and upper Darboux sums for f over $[a,b]$ such that
$$\lim(U_n - L_n) \;=\; 0.$$
By the various theorems of Section 32, we know that for every $n \in \mathbb{N}$,
$$L_n \;\leq\; L(f) \;\leq\; U(f) \;\leq\; U_n.$$
It follows that
$$0 \;\leq\; (U(f) - L(f)) \;\leq\; (U_n - L_n).$$
Since $\lim(U_n - L_n) = 0$, it follows by the Squeeze Theorem that $U(f) - L(f) = 0$. But this forces $L(f) = U(f)$, so that by definition, f is integrable on $[a,b]$. Thus $L(f) = U(f) = \int_a^b f$, and the inequality above is replaced by
$$L_n \;\leq\; \int_a^b f \;\leq\; U_n.$$
Thus for every n,
$$0 \;\leq\; (\int_a^b f - L_n) \;\leq\; (U_n - L_n).$$
Since $\lim(U_n - L_n) = 0$, it follows by the Squeeze Theorem that $\lim(\int_a^b f - L_n) = 0$. Thus $\lim\ L_n \;=\; \int_a^b f$. In a similar way, for every n,
$$0 \;\leq\; (U_n - \int_a^b f) \;\leq\; (U_n - L_n).$$
Since $\lim(U_n - L_n) = 0$, it follows by the Squeeze Theorem that $\lim(U_n - \int_a^b f) = 0$. Thus $\lim\ U_n \;=\; \int_a^b f$.

16. (p. 260, 33.4) Give an example of a function f defined on $[0,1]$ that is *not* integrable on $[0,1]$ but such that $|f|$ is integrable on $[0,1]$.
Define f on $[0,1]$ by
$$f(x) = \begin{cases} 1 & x \;\; \text{rational}, \\ -1 & x \text{ irrational}. \end{cases}$$
Let $\mathcal{P} = \{t_0 = 0 < t_1 < t_2 < \cdots < t_n = 1\}$ be an arbitrary partition of $[0,1]$. Then for all k going from 1 to n, using the density of irrational numbers
$$\inf\{f(x) : t_{k-1} \leq x \leq t_k\} = -1.$$

Similarly, using the density of rational numbers,

$$\sup\,\{f(x) : t_{k-1} \leq x \leq t_k\} = 1.$$

From this we deduce that for any partition $\mathcal{P}$ of $[0,1]$, $L(f,\mathcal{P}) = -1$ and $U(f,\mathcal{P}) = 1$. It follows that

$$U(f) \;=\; \inf\,\{U(f,\mathcal{P}\} \;=\; 1$$

and

$$L(f) = \sup\,\{L(f,\mathcal{P}\} \;=\; -1.$$

Since $L(f) = -1 < 1 = U(f),\;\; f$ is not integrable on $[0,1]$. On the other hand $|f(x)| = 1$ for all $x \in [0,1]$. Thus $|f|$ is a constant function, and constant functions are always integrable over any bounded interval.

17. (p. 190, 25.3) Let $f_n(x) = \frac{n+\cos x}{2n+\sin^2 x}$ for all real numbers x, and for $n \in \mathbb{N}$. Show that the sequence (f_n) converges uniformly on $\mathbb{R}$ to a limit function f. Then calculate

$$\lim_{n\to\infty}\int_2^7 f_n(x)dx.$$

(Follow the hint in the book!)
We consider the sequence of functions $f_n(x) = \frac{n+\cos x}{2n+\sin^2 x}$ defined on $\mathbb{R}$. We note $f_n(x) = \frac{1+(\cos x)/n}{2+(\sin^2 x)/n}$ for all $n \in \mathbb{N}$. We calculate

$$|f_n(x) - \frac{1}{2}| = |\frac{2+2(\cos x)/n}{2[2+(\sin^2 x)/n]} - \frac{2+(\sin^2 x)/n}{2[2+(\sin^2 x)/n]}|$$

$$|\frac{2(\cos x)/n - (\sin^2 x)/n}{2[2+(\sin^2 x)/n]}| \;\leq\; |\frac{2/n+1/n}{2\cdot 2}| \;=\; \frac{3}{4n}.$$

Let $\epsilon > 0$ be given. If $n > N = \frac{3\epsilon}{4}$, then

$$|f_n(x) - \frac{1}{2}| \;\leq\; \frac{3}{4n} \;<\; \epsilon,\; \forall\, x \;\in\; \mathbb{R}.$$

We note that N depends only on ϵ and not on x. It follows that (f_n) converges uniformly to the constant function $f(x) = \frac{1}{2}$ on $\mathbb{R}$. Hence (f_n) converges uniformly to $\frac{1}{2}$ on the interval $[2,7]$. Each function f_n is integrable on $[2,7]$ since each is continuous on $[2,7]$. By the theorem relating uniform convergence and integration, we obtain:

$$\lim_{n\to\infty}\int_2^7 \frac{1+(\cos x)/n}{2+(\sin^2 x)/n}dx \;=\; \int_2^7 \lim_{n\to\infty}\frac{1+(\cos x)/n}{2+(\sin^2 x)/n}dx$$

$$=\; \int_2^7 \frac{1}{2}\,dx \;=\; \frac{5}{2}.$$