University of Colorado
Department of Mathematics

2007/2008 Semester 1 | Math 4310 Introduction to Analysis | Midterm 2 Review

1. For each of the following statements, either prove that it is true, or give a counterexample to show that it is false.
   (a) If a sequence is monotone, then every one of its subsequences is monotone.
   (b) If a sequence is not monotone, then every one of its subsequences is not monotone.
   (c) If a sequence is unbounded, then every one of its subsequences is unbounded.
   (d) If a sequence is divergent, then it cannot have a convergent subsequence.
   (e) If a sequence tends to $+\infty$, then it cannot have a converegent subsequence.
   (f) If a sequence is unbounded, then it cannot have a convergent subsequence.
   (g) If $f : S \to \mathbb{R}$ is continuous on S, then f is uniformly continuous on S.
   (h) If $f : S \to \mathbb{R}$ is uniformly continuous on S, then f is continuous on S.

2. Let (s_n) be a convergent sequence of real numbers with limit not equal to 0. Define a sequence (t_n) by the formula

$$t_n \;=\; s_n + (-1)^n s_n, \text{ for all} n \in \mathbb{N}.$$

   (a) Show that (t_n) is divergent.
   (b) Show by any method that (t_n) has a convergent subsequence.

3. Show that if $\lim \frac{a_n}{n} = L$, where $L > 0$, then $\lim a_n = +\infty$.

4. Define a sequence (x_n) by

$$x_1 = 1, \;\; x_2 = 2, \;\; \text{and} \;\; x_{n+1} = x_n + \frac{1}{2^{n-1}}, \; n \geq 2, \; n \in \mathbb{N}.$$

   (a) Prove that (x_n) is a Cauchy sequence and hence convergent.
   (b) Find the limit of (x_n).

5. Let (s_n) be a sequence of real numbers. For each of the following statements, either prove that it is true, or give a counterexample to show that it is false.
   (a) If $\limsup s_n = 0$, then $\limsup |s_n| = 0$.

(b) If $\limsup|s_n| = 0$, then $\limsup s_n = 0$.

(c) If $\limsup|s_n| = 5$, then $\limsup s_n = 5$.

(d) If $\limsup|s_n| = 5$, then (s_n) is bounded.

6. Let (a_n) and (b_n) be sequences of real numbers. If $\sum_{n=1}^{\infty} a_n$ convereges absolutely, and (b_n) is bounded, prove that $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely.

7. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(x) < 0$ for each $x \in [0, 1]$. Prove that there exists a number $\beta < 0$ such that $f(x) \leq \beta$ for all $x \in [0, 1]$.

8. Show that the equation $x = [\cos x]^2$ has a solution in the interval $[0, \frac{\pi}{2}]$.

9. Let f be continuous on the interval $[0, 4]$ and suppose $f(0) = f(4)$. Prove that there exists a point $c \in [0, 2]$ such that $f(c) = f(c + 2)$. [Hint: Consider the function $g(x) = f(x) - f(x + 2)$ defined on $[0, 2]$.]

10. (a) Let f be defined on all $x \in \mathbb{R}\backslash\{2\}$ by

$$f(x) = \frac{x^2 + x - 6}{x - 2}.$$

Can f be defined at $x = 2$ in such a way that f is continuous at 2?

(b) Let f be defined on all $x \in \mathbb{R}\backslash\{1\}$ by

$$f(x) = \frac{x^2 - x - 2}{x - 1}.$$

Can f be defined at $x = 1$ in such a way that f is continuous at 1?

(c) Let f be defined on $(0, \frac{1}{\pi}]$ by

$$f(x) = x \sin \frac{1}{x}.$$

Can f be defined at $x = 0$ in such a way that f is continuous on $[0, \frac{1}{\pi}]$?

(d) Is $f(x) = x \sin \frac{1}{x}$ uniformly continuous on $(0, \frac{1}{\pi}]$?

11. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at c and suppose $f(c) > 0$. Show that there exists $\delta > 0$ such that $f(x) > 0$ whenever $|x - c| < \delta$.

12. Show that $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[1, \infty)$ but is not uniformly continuous on $(0, \infty)$.

13. (a) Show that if $f : J \to \mathbb{R}$ is uniformly continuous on J and (x_n) is a Cauchy sequence in J, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.

(b) Use the result just above to prove that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

14. (a) Is $f(x) = x \sin \frac{1}{x}$ uniformly continuous on $[\frac{1}{\pi}, \infty)$? Either prove the answer is yes or explain why not.

(b) Is $g(x) = x^3 \sin \frac{1}{x}$ uniformly continuous on $[\frac{1}{\pi}, \infty)$? Either prove the answer is yes or explain why not.